A STUDY OF SOME LINEAR AND NONLINEAR TIMOSHENKO BEAM MODELS

BY

SHADI MOHAMMAD AL-OMARI

A Dissertation Presented to the
DEANSHIP OF GRADUATE STUDIES

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the
Requirements for the Degree of

DOCTOR OF PHILOSOPHY

In

MATHEMATICS

May, 2017

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS
DHAHRAN 31261, SAUDI ARABIA

DEANSHIP OF GRADUATE STUDIES

This thesis, written by **SHADI MOHAMMAD AL-OMARI** under the direction of his thesis adviser and approved by his thesis committee, has been presented to and accepted by the Dean of Graduate Studies, in partial fulfillment of the requirements for the degree of **DOCTOR OF PHILOSOPHY IN MATHEMATICS** .

Dissertation Committee

Dr. Fiazud Din Zaman (Adviser)

Dr. Ashfaque H. Bokhari (Co-adviser)

Dr. Bilal Chanane (Member)

Dr. Faisal Fairag (Member)

Dr. Ahmet Z. Sahin (Member)

Dr. Husain Al-Attas
Department Chairman

Dr. Salam A. Zummo
Dean of Graduate Studies

عمادة الدراسات العليا
DEANSHIP OF GRADUATE STUDIES

Date 8/8/2017

To my passed away father, to my mother, my wife, my sons, my brothers and sisters.

To my brother, teacher and friend Dr. Ahmad Dweik.

ACKNOWLEDGMENTS

First and above all, I praise and glory ALLAH almightily who taught man through the pen. And peace and blessing of ALLAH be upon his last messenger Mohammad (Asllallah-Alihe-Wasallam), who brought the absolute truth to mankind.

Foremost acknowledgments are due to King Fahd University of Petroleum and Minerals that gave me the opportunity to work on my Ph.D program.

I express my sincere thanks and appreciation to my advisors Prof. F. D. Zaman and Prof. A. H. Bokhari for their guidance and encouragement throughout this research work.

Besides my advisors, I would like to thank the members of my thesis committee for their encouragement and valuable comments.

I extend my deepest gratitude to the man who always support me without anything in return and for his advices and his friendly assistance with various problems all the time, especially for his help with the paperwork. Dr. Ahmad Dweik thanks for all you've done for me.

Finally, my affectionate gratitude and appreciation go to my wife MAHA for unequivocal supporting throughout my study.

TABLE OF CONTENTS

ACKNOWLEDGEMENT **v**

LIST OF TABLES **ix**

LIST OF FIGURES **x**

ABSTRACT (ENGLISH) **xii**

ABSTRACT (ARABIC) **xiii**

INTRODUCTION **1**

CHAPTER 1 PRELIMINARIES **11**

1.1 Introduction 11

1.2 Lie Group of transformations 11

1.2.1 Groups 12

1.2.2 Group of transformations 13

1.2.3 One parameter Lie group of transformations 14

1.2.4 Examples 14

1.3 Infinitesimal transformations 15

1.3.1 Lie's First fundamental theorem and examples 16

1.3.2 Infinitesimal generators 17

1.3.3 Canonical Coordinates 22

1.4 Prolongation 25

1.5 Invariance 27

1.5.1 Invariant functions 27
1.5.2 Invariance of a PDE 28
1.6 r-Parameter Lie groups of transformations 29
1.7 Lie algebras and Structure constants 30
1.7.1 Lie algebras 30
1.7.2 Lie's second and third fundamental theorems 32
1.7.3 Solvable Lie algebra 33
1.7.4 Illustrative examples 34
1.8 One-dimensional optimal system 37

CHAPTER 2 HINGED- HINGED LINEAR TIMOSHENKO BEAM SYSTEM 39
2.1 Introduction 40
2.2 Mathematical analysis 44
2.2.1 Solution for the transverse displacement $\varphi(x,t)$ 46
2.2.2 Solution for the rotational angle $\psi(x,t)$ 50
2.3 Results and discussion 54
2.4 Conclusion 62
2.5 Finite Fourier sine and cosine transform formulas 63

CHAPTER 3 HINGED- HINGED LINEAR TIMOSHENKO BEAM SYSTEM WITH TWO WEAK DAMPING 65
3.1 Introduction 66
3.2 Mathematical analysis 70
3.3 Results and discussion 82
3.4 Conclusion 88
3.5 General Formula of Quartic Polynomial 90

CHAPTER 4 DAMPED TIMOSHENKO BEAM WITH NON-LINEAR ROTATIONAL MOMENT 92
4.1 Introduction 93

4.2 Complete Lie group classification 94
4.2.1 Linear rotational moment $\chi(\psi_x) = b\psi_x + \gamma$ 96
4.2.2 Non-Linear rotational moment $\chi_{\psi_x\psi_x} \neq 0$ 97
4.3 One-dimensional optimal system 103
4.3.1 Optimal system for the case $d^2 - 4k\rho_2 = 0$ 104
4.3.2 Optimal system for the case $d^2 - 4k\rho_2 = \lambda^2$ 109
4.3.3 Optimal system for the case $d^2 - 4k\rho_2 = -\mu^2$ 115
4.4 Optimal reductions and invariant solutions 119
4.5 Discussion and concluding remarks 127

CHAPTER 5 TIMOSHENKO BEAM WITH NON-LINEAR WEAK DAMPING 128

5.1 Introduction 129
5.2 Invariants in the adjoint representations 131
5.3 Lie's infinitesimal analysis 133
5.4 One-dimensional optimal system 139
5.5 Optimal reductions and invariant solutions 147
5.6 Discussion and concluding remarks 153

CHAPTER 6 CONCLUSIONS AND FUTURE WORK 155

REFERENCES 161

VITAE 173

LIST OF TABLES

1.1 Some known groups with their corresponding infinitesimals 19

1.2 Commutator Table 35

1.3 Commutator Table 36

2.1 Properties of Aluminum and Polycarbonate 54

3.1 Properties used in the simulations 80

3.2 Natural and damping frequencies of concrete and steel beams due to flexural and torsional damping. The cross-section of the beam modifies the damping factor of the beams. 81

4.1 The adjoint table corresponding to commutators (4.3.1) 104

4.2 The adjoint table corresponding to commutators (4.3.5) 109

4.3 The adjoint table corresponding to commutators (4.3.9) 115

4.4 Reduction using one-dimensional optimal system (4.3.3) 124

4.5 Reduction using one-dimensional optimal system (4.3.7) with . . 125

4.6 Reduction using one-dimensional optimal system (4.3.11) 126

5.1 Adjoint table 140

5.2 Reductions using one-dimensional optimal system (5.4.3) 151

5.3 Invariant conditions and theirs invariant reductions 152

LIST OF FIGURES

2.1 Temporal variation of displacement and angular rotation of aluminum beam. 57
2.2 Temporal variation of displacement and angular rotation of polycarbonate beam. 58
2.3 Variation of displacement and angular rotation along the x-axis for aluminum beam. 60
2.4 Variation of displacement and angular rotation along the x-axis for polycarbonate beam. 61

3.1 Flexural and rotational oscillations of hinged-hinged concrete beam for different damping conditions. The cross-sectional size of the rectangular beam is width (b) = 0.01 m and height (h) = 0.1 m. The length of the beam is 1 m. d1 (0.0233) is the flexural damping factor and d2 (5.24×10^{-6}) is the torsional damping factor of concrete. x- axis locations represent the locations along the beam length. 84
3.2 Flexural and rotational oscillations of hinged-hinged concrete beam for different damping conditions. The cross-sectional size of the rectangular beam is width (b) = 0.02 m and height (h) = 0.05 m. The length of the beam is 1 m. d1 (0.012) is the flexural damping factor and d2 (5.01×10^{-6}) is the torsional damping factor of concrete. x- axis locations represent the locations along the beam length. 85

3.3 Flexural and rotational oscillations of hinged-hinged steel beam for different damping conditions. The cross-sectional size of the rectangular beam is width (b) = 0.01 m and height (h) = 0.1 m. The length of the beam is 1 m. The flexural damping factor is d1 (0.0011) and the torsional damping factor is d2 (3.55×10^{-7}) of steel beam. x- axis locations represent the locations along the beam length. 87

3.4 Flexural and rotational oscillations of hinged-hinged steel beam for different damping conditions. The cross-sectional size of the rectangular beam is width (b) = 0.02 m and height (h) = 0.05 m. The length of the beam is 1 m. The flexural damping factor is d1 (0.00053) and the torsional damping factor is d2 ($3.39 \times 10-7$) of steel beam. x- axis locations represent the locations along the beam length. 88

4.1 Tree diagram 107

4.2 Tree diagram 118

5.1 Tree diagram 144

THESIS ABSTRACT

NAME:	Shadi Mohammad Al-Omari
TITLE OF STUDY:	A study of some linear and nonlinear Timoshenko beam models
MAJOR FIELD:	Mathematics
DATE OF DEGREE:	May, 2017

Four models arising from linear and nonlinear Timoshenko beam theory have been studied. The exact solution for the linear cases which represents hinged-hinged damped and undamped Timoshenko beam with suitable initial conditions is obtained. Two examples of Aluminum and Polycarbonate beam materials are given for undamped model, and two examples of Concrete and Steel beam are given for damped model. Two nonlinear Timoshenko beam models are considered, one with nonlinear rotational moment and the other with nonlinear weak damping. A complete Lie group classification, adjoint representation, one-dimensional optimal system and the optimal reduction with corresponding invariant solutions for both nonlinear cases are found.

INTRODUCTION

A differential equation (DE) is an equation that contain function with its derivative. The function usually represents physical phenomena or quantities, while the derivatives represent the rates of change of these quantities. In this sense DEs play a significant role in applied mathematics, biology, engineering, physics and almost all science fields.

One of the most famous and powerful methods for studying and finding exact solutions of DEs is the symmetry analysis method [5, 6, 33, 34, 35, 50, 51, 53], which is also called group analysis. Symmetry analysis established by the Norwegian mathematician Sophus Lie (Figure 1), during the period 1872-1899 [43, 44]

Sophus Lie (Dec. 1842 - Feb. 1899)

One can refer to [36, 86] for a detailed historical description.

In simple words, a symmetry in mathematics means that a shape exactly remains the same when a change or transformation takes place, like turn, flip or slide. For instance: the rotation of a circle in any direction about its center is a symmetry. As a consequence the symmetry of a DE is a transformation that maps any solution of the DE to another solution. One of such transformations was introduced by Lie in the form of a point transformation. A Lie point symmetry is characterized by an infinitesimal transformation that leaves the DE invariant under the transformation of the variables (independent and dependent).

Nowadays, we know many applications of Lie symmetry analysis for DE's, such as:

1. Reduction of the order of the DE.

2. Generating a new solutions from known ones.

3. Reduction in the number of independent variables when Lie symmetry method apply on partial differential equations (PDEs).

4. Finding invariant solutions.

5. Construction of invariant solutions for boundary value problems(BVPs).

6. Finding conservation laws.

7. Detection of linearizing transformations of DE's.

For many other applications of Lie group analysis or the Lie analysis method the reader should refer to [3, 6, 51, 53].

It is important to mention that studying Lie symmetry analysis and its applications to any mathematical model containing a single or system of DEs involves tedious computations. Even the calculation of symmetry group of a humble systems of DEs is prone to fail if one does it with pencil and paper. Many software packages are available in this area such as [10, 18, 29, 67]. In this work, the "Janet" [4], and "SADE" [65] Maple packages and many other simple codes written in maple and mathematica are used to help us in the tedious computations we faced.

In this dissertation, we use Lie symmetry approach to study some models from the transverse vibration problem of beam which is essential in many engineering applications. Our study centers on Timoshenko beam model, which were introduced by Stepan Timoshenko in 1921, where the problem of transverse vibrating beam was formulated in terms of PDEs taking shear and rotation effects in to account. Here we will present a brief historical review for each one of the four models of transverse vibrating problem which are Euler-Bernoulli, Rayleigh, shear and Timoshenko beam model.

Traill-Nash and Collar [81] considered different types of end conditions for the four models and derived the frequency equations corresponding to each end condition. One of the most complete studies of the four transverse vibrating models was done by Han et. al. [28] where the equations of motion for each model were obtained, and the frequency equations are obtained for free-free, clamped-clamped, hinged-hinged and clamped-free end conditions.

Some of the early studies were based upon the Euler-Bernoulli model which takes into account bending moment and lateral displacement, date back to the 18th century. Jacob Bernoulli and Daniel Bernoulli noted that bending moment is proportional to the curvature of elastic beam at any fixed point. They formulated the equation of motion of vibrating beam taking bending moment only into account. Later Leonhard Euler [78] made many advances on Bernoulli's theory when he was studying the effect of different loading conditions on the elastic curves of the beam.

From Lie symmetry point of view, one of the earliest studies for the Euler-Bernoulli beam model was done by Bokhari et. al. [7]. They considered the static Euler-Bernoulli model of the form

$$\frac{d^2}{dx^2}(EI\frac{d^2y}{dx^2}) = f(y),$$

where $f(y(\cdot))$ is the applied load function. They presented a complete group classification for different type of applied load function, and through Noether integral they derived a two-parameter family of exact solutions. Fatima et. al. [24] obtained a new two and three parameter families of exact solutions of the static Euler-Bernoulli equation above. Wafo Soh [70] solved the equivalence problem of dynamic Euler-Bernoulli equation

$$\frac{\partial^2}{\partial x^2}(f(x)\frac{\partial^2 u}{\partial t^2}) + m(x)\frac{\partial^2 u}{\partial t^2} = 0, \qquad t > 0, \qquad 0 < x < L,$$

using Lie symmetry analysis, where $f(x) > 0$ is the flexural rigidity, $m(x) > 0$ is the linear mass density. Recently, Naz and Mohamed [48] added an applied load function to the dynamic Euler-Bernoulli model in the work [70] and provided a complete study of Lie group classification for different types of linear mass density and applied load function. Ozkaya and Pakdemirili [56] considered the transverse vibration of a beam moving with time-dependent velocity and used symmetry analysis to obtain approximate solution for exponentially decaying problem.

The Rayleigh beam theory suggested by Strultt and Rayleigh [73, 74] considered as an extension of Euler-Bernoulli beam theory since the effect of rotation of the cross section of the beam is included in this theory. Davies [15] also studied the effect of rotary inertia on a free-fixed end conditions of a beam.

Another extension of Euler-Bernoulli beam theory is by adding the effect of transverse shear strain. This effect is negligible for thin beams but it is significant when thick beams are consider. Shear beam theory [1] does not fit our purpose of improved Euler-Bernoulli theory since the bending effect which is the most important factor is excluded in this theory.

Timoshenko beam model which is adapted in this work was developed by Stephen Timoshenko [79, 80] who proposed taking into consideration the shear as well as the rotation effects which proved to be suitable for non-slender beams and high frequency vibrations. Considering rotation and shear effects makes Timoshenko beam theory a major improvement of the Euler-Bernoulli theory. Trill-Nash and Collar [81], Haung [30] and Kruszewski [41] have obtained the frequency equations with the mode shapes corresponding to different boundary conditions.

From symmetry point of view Vassilev and Djondjorov [84] study the Lie point symmetry and the conservation laws of Euler-Bernoulli and Timoshenko beam theories. Djondjorov in his paper [17] studied the invariant properties of Timoshenko beam system of the form

$$\rho I\phi_{tt} = EI\phi_{xx} + nGA(W_x - \phi),$$

$$\rho AW_{tt} = nGA(W_{xx} - \phi_x) + q(x,t),$$

where $W(x,t)$ is the transverse displacement and $\phi(x,t)$ is the rotational angle of the beam. The most general invariant solution with arbitrary one-parameter symmetry group is obtained for the above system.

Revera et. al. [63] study the global stability for the following damped Timoshenko beam system with non-linear rotational moment

$$\rho_1\phi_{tt} - k(\phi_x + \psi)_x = 0,$$

$$\rho_2\psi_{tt} - (\chi(\psi_x))_x + k(\phi_x + \psi) + d\psi_t = 0,$$

which we will consider in chapter (4) of this dissertation from symmetry analysis standpoint. Mustafa and messaudi [47] and ftiha [23] studied the stability of the following Timoshenko system with non-linear damping term

$$\rho_1\phi_{tt} - k(\phi_x + \psi)_x = 0,$$

$$\rho_2\psi_{tt} - EI\psi_{xx} + k(\phi_x + \psi) + \chi(\psi_t) = 0,$$

where the functions ϕ and ψ depend on $(t, x) \in (0, \infty) \times (0, L)$ and model the vertical displacement and the rotational angle of the beam respectively. This system will be considered in chapter (5) of this work from symmetry point of view.

The following table provides the main difference between the four beam theories

The four beam models

Beam models	Bending moment	Transverse displacement	Shear deformation	Rotary inertia
Euler-Bernoulli	$\surd$	$\surd$	$\times$	$\times$
Rayleigh	$\surd$	$\surd$	$\times$	$\surd$
Shear	$\surd$	$\surd$	$\surd$	$\times$
Timoshenko	$\surd$	$\surd$	$\surd$	$\surd$

This dissertation contains the following six chapters:

In **Introduction** we present a brief history review of the four beam theories. In **chapter 1** we describe the basic concepts that are needed to study the linear and nonlinear Timoshenko beam models. We provide the fundamental notations of the symmetry analysis of differential equations, continuous group theory and optimal system. In **chapter 2** a hinged-hinged beam resembling the Timoshenko beam characteristics is presented incorporating the appropriate initial and boundary conditions. The finite Fourier transformation method is incorporated to solve the governing equations of momentum for the Timoshenko beam. The closed form solutions for the beam displacement and rotation are simulated for two different beam materials, namely aluminum and polycarbonate, having the square cross section. In **Chapter 3** analytical treatment of flexural and torsional characteristics of hinged-hinged Timoshenko system is carried out and the solution for the dynamic response of the beam due to external excitation is presented. In order to assess damping effect on flexural and torsional characteristics of hinged-hinged beam, the aspect ratio of the beam cross-section is changes while keeping the beam cross-sectional area constant. This arrangement provides the change of the second moment of an area and the damping factor of the beam. Two beam materials, namely concrete and steel, are incorporated in analysis to resemble the high and low damping materials. It is found that introducing the damping factor of the beam material for flexural motion increases logarithmic decay of flexural and torsional oscillations, despite torsional motion is considered to be elastic. Reducing the aspect ratio of the beam cross-section lowers logarithmic decay of the amplitude of flexural and torsional oscillations, which is more pronounced for the steel beam. In addition, this arrangement modifies the damping frequency of the oscillations. In **chapter 4** a non-linear Timoshenko system with frictional damping term in rotation angle is considered. The nonlinearity is due to the arbitrary dependence on the rotation moment. A Lie symmetry group classification of the arbitrary function of rotation moment is presented. In **chapter 5** we consider another non-linear class of Timoshenko system with frictional damping in rotation angle. The non-linearity is due to the arbitrary dependence on the rotational damping ψ_t. Lie symmetries and their Lie group transformations for Timoshenko system are presented, we also find all possible non-similar invariant conditions prescribed on invariant surfaces under symmetry transformations. As an application, we study a beam which is hinged at one end where a constant control torque is applied, and free at the other end where the linear control force is applied. Some conclusions and future directions of work we aim to pursue are presented in **chapter 6**.

CHAPTER 1

PRELIMINARIES

1.1 Introduction

This chapter describes briefly the basic concepts that are needed to study the linear and nonlinear Timoshenko beam models, which are wildly used in structure mechanics and civil engineering. We provide the fundamental notations of the symmetry analysis of differential equations, continuous group theory and optimal system. All of the proofs in this short review are omitted since these are available in the literature. One can refer to the standard books [5, 6, 34, 35, 33, 51, 72] for more details about Lie symmetry analysis of ordinary differential equations (ODEs), partial differential equations (PDEs) and its applications.

1.2 Lie Group of transformations

We present the concept of group, then we define group of transformations and a one-parameter Lie group of transformation with some illustrative examples.

1.2.1 Groups

Definition 1.2.1 *A binary operator on a set of elements is a map combines any two elements in the set to form another element belong to the same set.*

Definition 1.2.2 *[51] A group (G,$*$) is a non empty set G together with a group operation $*$, such that for any two elements α and β of G we have $\alpha * \beta$ is also an element if G. The group operation $*$ must satisfy the following axioms:*

*(1)**Associativity**. If α,β and γ are elements of G, then*

$$\alpha * (\beta * \gamma) = (\alpha * \beta) * \gamma.$$

*(2)**Identity Element**. In G there exist a distinct element e known as the identity element such that for any element α in G*

$$\alpha * e = \alpha = e * \alpha.$$

*(3) **Inverse**. For every α in G there exist an element α^{-1} known as inverse of α, with the property*

$$\alpha * \alpha^{-1} = \alpha^{-1} * \alpha = e$$

*where e is the identity element of G with respect to the binary operator *.*

Definition 1.2.3 *G is an Abelian group if it is commutative in addition to the above properties. i.e.*

$$\text{For all } \alpha,\ \beta \text{ in } G,\ \alpha * \beta = \beta * \alpha$$

Definition 1.2.4 *A subset H of G is called subgroup of G under the binary operator $*$, if H is also a group under $*$. The trivial subgroup of G is the subgroup which consist just the identity element e. A proper subgroup of G is a proper subset H of G ($H \neq G$) which satisfy the group properties.*

Example 1.2.1 *The following are examples of groups:*

(1) *G is the set of all nonzero real number $\mathbb{R}\backslash\{0\}$ with multiplication $\alpha * \beta = \alpha\beta$. Here, $e = 1$ and $\alpha^{-1} = \frac{1}{\alpha}$.*

(2) *G is the set $\mathbb{Z}_n$ of congruence classes modulo n with addition $[\alpha]*[\beta] = [\alpha+\beta]$. Here, $e = [0]$ and $[\alpha]^{-1} = [-\alpha]$.*

(3) *G is the set of positive real number $\mathbb{R}^+$ with $\alpha * \beta = 2\alpha\beta$. Here, $e = \frac{1}{2}$ and $\alpha^{-1} = \frac{1}{4\alpha}$.*

1.2.2 Group of transformations

Definition 1.2.5 *Consider a domain $D \subseteq \mathbb{R}^n$ and a subset $S \subseteq \mathbb{R}$. The set of transformations*

$$x^\star = F(x, a), \quad F : D \times S \longrightarrow D$$

defined for all $x \in D$ and depending on the parameter $a \in S \subseteq \mathbb{R}$ with the operator $*$, forms a one-parameter group of transformation on D if:

(1) For each $a \in S$, the transformations F are one-to-one and onto D;

(2) The subset S with the operator $*$ is a group with identity e;

(3) $F(x,e) = x$, *for all* $x \in D$;

(4) $F(F(x,a),b) = F(x, a * b)$, *for all* $x \in D$ *and for all* $a, b \in S$.

1.2.3 One parameter Lie group of transformations

Definition 1.2.6 *The group of transformations in Definition 1.2.5, with the operation* $*$, *is said to be one-parameter Lie group of transformations, if it satisfies the conditions in the previous definition (1.2.5) in addition to the following:*

(1) The parameter a *is continuous, i.e.* S *is an interval in* $\mathbb{R}$ *containing the identity element;*

(2) $F \in C^\infty$ *(i.e.* F *is infinitely differentiable) with respect to* $x \in D$, *and analytic function of* $a \in S$;

(3) $*(a,b)$ *is an analytic function of* a *and* b, *for* $a, b \in S$.

1.2.4 Examples

The following are examples of a one-parameter Lie group of transformations

Example 1.2.2 *Group of translation in the plane*

$$\begin{aligned} x^\star = & \quad x + a, \\ y^\star = & \quad y, \qquad a \in \mathbb{R} \end{aligned}$$

with the operation $*(a,b) = a + b$, *and identity* $e = 0$.

Example 1.2.3 *Group of scaling in the plane defined by*

$$x^\star = \ ax,$$
$$y^\star = \ a^2 y, \quad 0 < a < \infty$$

with the operation $*(a,b) = ab$*, and the identity element given by* $a = 1$*.*

Example 1.2.4 *Group of rotations*

$$x^\star = \ x \ cos(a) + y \ sin(a)$$
$$y^\star = \ y \ cos(a) - x \ sin(a)$$

with the operation $*(a,b) = a + b$*, and the identity element* $e = 0$*.*

Example 1.2.5 *Now, let us consider the transformation given by*

$$x^\star = \ -x$$
$$y^\star = \ -y.$$

This transformation does not form Lie group since $x^{\star\star} = x$*, and* $y^{\star\star} = y$*.*

1.3 Infinitesimal transformations

Expanding the one-parameter group of transformations $x^\star = F(x; a)$, with the identity $a = 0$, and the analytic binary operator $*$, in Taylor series in a and in neighborhood of $a = 0$, taking into account the third conditions in Definition 1.2.5, which gives $x^{\star}|_{a=0} = F(x;0) = x$, we get

$$\begin{aligned} x^{\star} &= F(x;a)\big|_{a=0} + a\tfrac{\partial F(x;a)}{\partial a}\big|_{a=0} + \tfrac{a^2}{2}\tfrac{\partial^2 F(x;a)}{\partial a^2}\big|_{a=0} + \ldots \\ &= x + a\tfrac{\partial F(x;a)}{\partial a}\big|_{a=0} + \mathcal{O}(a^2). \end{aligned}$$

Let $\frac{\partial x^{\star}}{\partial a}\big|_{a=0} \equiv \frac{\partial F(x;a)}{\partial a}\big|_{a=0} = \xi(x)$, then the transformation $x^{\star} = x + a\xi(x)$ is called the infinitesimal transformation of the Lie group of transformation, and the component $\xi(x)$ is called the infinitesimal of $x^{\star} = F(x;a)$.

1.3.1 Lie's First fundamental theorem and examples

The first fundamental theorem of Lie guarantees that the infinitesimal transformations include all what is needed for describing a one-parameter Lie group of transformations.

Theorem 1.3.1 *[5] There exists a parametrization $\tau(x)$ such that the Lie group of transformations $x^{\star} = F(x;a)$ is equivalent to the solution of the initial value problem for the system of first order differential equations*

$$\tfrac{dx^{\star}}{da} = \xi(x^{\star}), \qquad \textit{with} \qquad x^{\star}(0) = x.$$

Example 1.3.1 *The group of translations in the plane*

$$x^{\star} = x + a, \quad y^{\star} = y,$$

where $a \in \mathbb{R}$, with the law of composition given by $*(a,b) = a + b$, and $a^{-1} = -a$, is equivalent to the solution of the Initial Value Problem (IVP)

$$\tfrac{dx^\star}{da} = 1, \;\; \tfrac{dy^\star}{da} = 0 \qquad with \qquad x^\star \mid_{a=0}= x, \; y^\star \mid_{a=0}= y.$$

Example 1.3.2 *The group of scaling*

$$x^\star = (1+a)x, \quad y^\star = (1+a)^2 y,$$

where $0 < a < \infty$, with the law of composition given by $*(a,b) = a + b + ab$, and $a^{-1} = \frac{-a}{1+a}$, is equivalent to the solution of the IVP

$$\tfrac{dx^\star}{da} = \tfrac{x^\star}{1+a}, \;\; \tfrac{dy^\star}{da} = \tfrac{2y^\star}{1+a} \qquad with \qquad x^\star \mid_{a=0}= x, \; y^\star \mid_{a=0}= y.$$

1.3.2 Infinitesimal generators

For simplicity let us start with $x = (x_1, x_2) \in S \subset \mathbb{R}^2$. Consider a one-parameter group of transformation:

$$x_1^\star = F_1(x_1, x_2; a);$$

$$x_2^\star = F_2(x_1, x_2; a);$$

with

$$F_1(x_1, x_2; a) \mid_{a=0}= x_1, \qquad and \qquad F_2(x_1, x_2; a) \mid_{a=0}= x_2.$$

Clearly, this defines a curve $\Gamma(a) = (x_1^\star, x_2^\star) = (F_1(x_1, x_2; a), F_2(x_1, x_2; a))$, the tangent vector field $(\xi^1(x_1, x_2), \xi^2(x_1, x_2))$ can be written as first order differential operator

$$X = \xi^1(x_1, x_2)\frac{\partial}{\partial x_1} + \xi^2(x_1, x_2)\frac{\partial}{\partial x_2}, \tag{1.3.1}$$

This differential operator is called the infinitesimal generator of a one-parameter group of transformations, where

$$\xi^1(x_1, x_2) = \tfrac{\partial F_1(x_1,x_2;a)}{\partial a} \mid_{a=0},$$

$$\xi^2(x_1, x_2) = \tfrac{\partial F_2(x_1,x_2;a)}{\partial a} \mid_{a=0} .$$

In general, for $x = (x_1, x_2, ..., x_n) \in S \subset \mathbb{R}^n$ we have the following:

Definition 1.3.1 *The infinitesimal generator of the one parameter Lie group of transformations $x^\star = F(x; a)$ is the differential operator*

$$X = \xi(x) \cdot \nabla = \sum_{i=1}^{n} \xi_i(x) \frac{\partial}{\partial x_i}, \tag{1.3.2}$$

where $\xi_i = \frac{\partial x_i^\star}{\partial a} \mid_{a=0}$, and $\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_n})$ is the gradient operator.

It is clear now, how to evaluate the infinitesimal generator corresponding to a group of transformations. For converse, suppose one has the infinitesimal generator

$$X = \xi^1(x_1, x_2)\frac{\partial}{\partial x_1} + \xi^2(x_1, x_2)\frac{\partial}{\partial x_2},$$

then the associated group of transformation

$$x_1^\star = F_1(x_1, x_2; a), x_2^\star = F_2(x_1, x_2; a)$$

with

$$F_1(x_1, x_2; 0) = x_1, \qquad F_2(x_1, x_2; 0) = x_2,$$

can be found by solving the IVP

$$\frac{dx_1^\star}{da} = \xi^1(x_1, x_2), \quad \frac{dx_2^\star}{da} = \xi^2(x_1, x_2),$$

with $x_1^\star \mid_{a=0} = x_1$, and $x_2^\star \mid_{a=0} = x_2$.

The following table presents some known group transformations in the (x, y)-plane with their corresponding infinitesimal generators.

Table 1.1: Some known groups with their corresponding infinitesimals

Action	Group	Infinitesimal generator
Translation in x	$x^\star = x + a$ $y^\star = y$	$\frac{\partial}{\partial x}$,
Translation in y	$x^\star = x$ $y^\star = y + a$	$\frac{\partial}{\partial y}$,
Translation in both	$x^\star = x + a$ $y^\star = y + a$	$\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$,
Scaling in x	$x^\star = e^a x$ $y^\star = y$	$x\frac{\partial}{\partial x}$,
Scaling in y	$x^\star = x$ $y^\star = e^a y$	$y\frac{\partial}{\partial y}$,
Scaling in both	$x^\star = e^{c_1 a} x$ $y^\star = e^{c_2 a} y$	$c_1 x\frac{\partial}{\partial x} + c_2 y\frac{\partial}{\partial y}$,
Rotation	$x^\star = xcos(a) - ysin(a)$ $y^\star = xsin(a) + ycos(a)$	$-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$,

Lie's First theorem implies that a one-parameter Lie group of transformations is equivalent to its corresponding infinitesimal operator. Using the infinitesimal operator one can represent the group transforation in terms of a Taylor series in the parameter a (near $a = 0$) to be equivalent to the following exponential map which is called the Lie series:

$$\begin{aligned} x^\star &= e^{aX}x = x + aXx + \tfrac{a^2}{2}X^2x + \tfrac{a^3}{6}X^3x + ... \\ &= \textstyle\sum_{k=0}^{\infty} \tfrac{a^k}{k!}X^k x, \end{aligned}$$

where, X is the infinitesimal operator in equation (1.3.1), $X^0x = x$ and $X^kx = X(X^{k-1}x)$.

Remark 1.3.1 *[5] There are two ways to find explicitly a one-parameter Lie group of transformations from its infinitesimal transformation:*

(A) Express the group in terms of the power series $x^\star = \sum_{k=0}^{\infty} \frac{a^k}{k!}X^k x$, *which is called Lie series, that is developed from the infinitesimal generator (1.3.1) corresponding to the infinitesimal transformation;*

(B) Solve the IVP, $\frac{dx^\star}{da} = \xi(x^\star)$ *with* $x^\star\,|_{a=0} = x$, *through explicitly finding the general solution of the system of first order differential equations* $\frac{dx^\star}{da} = \xi(x^\star)$.

Example 1.3.3 *Consider the following generator in the plane*

$$X = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y},$$

the Lie series corresponding to this infinitesimal generator is given by

$$(x^\star, y^\star) = (e^{aX}x, e^{aX}y),$$

where, $X^0(x) = x$, $X^1(x) = x^2$, $X^2(x) = 2!x^3$, $X^3(x) = 3!x^4$ *and so on.*

Note that

$$X^k(x) = k!x^{k+1},$$

since by induction

$$X^{k+1}(x) = X(k!x^{k+1}) = (k+1)!x^2x^k = (k+1)!x^{k+2}.$$

On the other hand, $X^0(y) = y$, $X^1(y) = xy$, $X^2(y) = 2!x^2y$, $X^3(y) = 3!x^3y$, *etc.*

Also,

$$X^k(y) = k!x^ky,$$

since similarly by induction

$$X^{k+1}(y) = k!X(yx^k) = k![kx^{k+1}y + x^k(xy)] = (k+1)!x^{k+1}y.$$

Now, substituting in the exponential map implies

$$
\begin{aligned}
x^\star &= e^{aX}x = \textstyle\sum_{k=0}^{\infty} \frac{a^k}{k!} X^k x \\
&= X^0(x) + aX^1(x) + \tfrac{a^2}{2!}X^2(x) + \tfrac{a^3}{3!}X^3(x) + ... \\
&= x + ax^2 + a^2x^3 + a^3x^4 + ... \\
&= x(1 + ax + a^2x^2 + a^3x^3 + ...) \\
&= \tfrac{x}{1-ax}, \qquad for \quad |ax| < 1.
\end{aligned}
$$

Likewise,

$$
\begin{aligned}
y^\star &= e^{aX}y = \textstyle\sum_{k=0}^{\infty} \frac{a^k}{k!} X^k y \\
&= X^0(y) + aX^1(y) + \tfrac{a^2}{2!}X^2(y) + \tfrac{a^3}{3!}X^3(y) + ... \\
&= y + axy + a^2x^2y + a^3x^3y + ... \\
&= y(1 + ax + a^2x^2 + a^3x^3 + ...) \\
&= \tfrac{y}{1-ax}, \qquad for \quad |ax| < 1.
\end{aligned}
$$

So, we arrive at the a one-parameter group of transformations of the form

$$
x^\star = \frac{x}{1-ax}, \qquad y^\star = \frac{y}{1-ax}.
$$

1.3.3 Canonical Coordinates

In many situations, it is not easy to deal with differential equation in its given coordinates. A good choice in this cases may be obtained by changing to a new coordinate system that make the differential equation easier to solve. Therefore, one can change to the canonical coordinate $r(x,y), s(x,y)$ so that the point symmetry

$$x^\star = X(x,y;a) = x + a\xi(x,y) + \mathcal{O}(a^2),$$

$$y^\star = Y(x,y;a) = y + a\eta(x,y) + \mathcal{O}(a^2),$$

becomes the translation group

$$r^\star = r, \qquad s^\star = s + a.$$

To show how one can get this transformation consider the transformation

$$\Gamma_a : (r,s) \to (r^\star, s^\star) = (r, s+a),$$

in this transformation the tangent vector field at (r,s) when $a=0$ is given by

$$\tfrac{\partial r^\star}{\partial a}\big|_{a=0} = 0 \quad and \quad \tfrac{\partial s^\star}{\partial a}\big|_{a=0} = 1.$$

By chain rule we get

$$\frac{\partial r^\star}{\partial a}\bigg|_{a=0} = \frac{\partial r^\star}{\partial x}\frac{\partial x}{\partial a}\bigg|_{a=0} + \frac{\partial r^\star}{\partial y}\frac{\partial y}{\partial a}\bigg|_{a=0} = \xi(x,y)\frac{dr}{dx} + \eta(x,y)\frac{dr}{dy} = 0,$$

and

$$\frac{\partial s^\star}{\partial a}\bigg|_{a=0} = \frac{\partial s^\star}{\partial x}\frac{\partial x}{\partial a}\bigg|_{a=0} + \frac{\partial s^\star}{\partial y}\frac{\partial y}{\partial a}\bigg|_{a=0} = \xi(x,y)\frac{ds}{dx} + \eta(x,y)\frac{ds}{dy} = 1.$$

Both of those equations can be rewritten as

$$r_x\xi(x,y) + r_y\eta(x,y) = 0, \qquad i.e. \quad X_r = 0,$$

and

$$s_x\xi(x,y) + s_y\eta(x,y) = 0, \qquad i.e. \quad X_s = 1.$$

We can now solve these differential equations by using the method of characteristics.

Example 1.3.4 *Consider the following one-parameter group of scaling transformations*

$$x^\star = e^a x,$$

$$y^\star = e^{ak} y,$$

the corresponding infinitesimal generator is given by

$$X = x\frac{\partial}{\partial x} + ky\frac{\partial}{\partial y}.$$

To evaluate the canonical coordinate $r(x,y)$ *and* $s(x,y)$ *we use*

$$X_r = 0 \quad \Rightarrow \quad x\frac{\partial r}{\partial x} + ky\frac{\partial r}{\partial y} = 0.$$

The corresponding characteristic differential equation

$$\frac{dy}{dx} = \frac{ky}{x},$$

has solution of the form $r(x,y) = \frac{y}{x^k}$*. Similarly we get*

$$X_s = 1 \quad \Rightarrow \quad x\frac{\partial s}{\partial x} + ky\frac{\partial s}{\partial y} = 1,$$

has solution given by $s(x,y) = s(x)$ *satisfying* $\frac{ds}{dx} = \frac{1}{x}$*, having the solution* $s(x,y) = ln(x)$*. Hence, the canonical coordinates corresponding to scaling transformation are*

$$(r(x,y), s(x,y)) = (\frac{y}{x^k}, ln(x)).$$

1.4 Prolongation

In many situations we need to apply the infinitesimal criterion of a k^{th} order PDE or system of PDEs with n independent and m dependent variables. In such cases, when one need to check the invariant of system of PDEs of order k, as we will see later, the infinitesimal (1.3.1) on its current form does not work on system of PDEs. We need to extend the infinitesimal generator to contain all the independent variables, independent variables and the derivative of dependent variable of all order till k.

To find such prolongation formula of order k for the infinitesimal (1.3.1), consider the k^{th} order system of PDEs with n independent variables $x = (x_1, x_2, ..., x_n)$ and m dependent variables $u = (u^1, u^2, ..., u^m)$, on the form

$$\Gamma^\alpha(x, u, \partial u, \partial^2 u, ..., \partial^k u) = 0, \qquad \alpha = 1, 2, ..., k,$$

where $(\partial u, \partial^2 u, ..., \partial^k)$ denote the first, second, ..., k^{th} order partial derivative, such that $u_i^\alpha = \frac{\partial u^\alpha}{\partial x_i} = D_i(u^\alpha)$, and $u_{ij}^\alpha = \frac{\partial^2 u^\alpha}{\partial x_i \partial x_j} = D_i D_j(u^\alpha)$, and so on. Where D_i is the total derivative operator given by

$$D_i = \frac{\partial}{\partial x_i} + u_i^\mu \frac{\partial}{\partial u^\mu} + u_{ij}^\mu \frac{\partial}{\partial u_j^\mu} + ... + u_{ii_1i_2...i_n}^\mu \frac{\partial}{\partial u_{i_1i_2...i_n}^\mu} + ...$$

This extension deals with transformation in (x, u)-space to $(x, u, \partial u, \partial^2 u, ..., \partial^k)$-space, with the consideration that $\partial^k u$ denotes the components of all k^{th} order partial derivatives of u with respect to x_i.

Here, the k^{th} extended infinitesimal generator of (1.3.1) is given by

$$\begin{aligned} X^{(k)} &= \xi_i(x, u)\tfrac{\partial}{\partial x_i} + \eta^\mu(x, u)\tfrac{\partial}{\partial u^\mu} + \eta_i^{(1)\mu}(x, u, \partial u)\tfrac{\partial}{\partial u_i^\mu} + ... \\ &+ \eta_{i_1i_2...i_k}^{(k)\mu}(x, u, \partial u, \partial^2 u, ..., \partial^k u)\tfrac{\partial}{\partial u_{i_1i_2...i_k}^\mu}, \quad k \geq 1, \end{aligned} \tag{1.4.1}$$

where,

$$\eta^{(1)\mu} = D_i \eta^\mu - (D_i \xi_j) u_j^\mu,$$

and

$$\eta_{i_1i_2...i_k}^{(k)\mu} = D_{i_k} \eta_{i_1i_2...i_{k-1}}^{(k-1)\mu} - (D_{i_k} \xi_j) u_{i_1i_2...i_{k-1}j}^\mu,$$

with $i_p = 1, 2, ..., n$ for $p = 1, 2, ..., k$.

1.5 Invariance

Invariant of a quantity or a relation or a property means that it will not change after a certain class of transformations acts on it. The simplest example of invariance is the scalar function, also a circle is invariant under any rotation. Invariant functions and invariant PDEs are important concepts for studying symmetry analysis. For more details the reader should refer to [52].

1.5.1 Invariant functions

Let $H(\cdot)$ be an infinitly differentable function (i.e. $H(\cdot) \in C^{\infty}$). $H(\cdot)$ is said to be invariant function of the Lie group of translation $x^{\star} = F(x; a)$ if and only if, for any group translation $x^{\star} = F(x; a)$

$$H(x^{\star}) = H(x).$$

If this condition holds, then $H(x)$ is invariant under $x^{\star} = F(x; a)$.

The concept of invariance of a function can be characterized simply using the infinitesimal generator as in the following theorem.

Theorem 1.5.1 *[5] $H(x)$ is invariant under $x^{\star} = F(x; a)$ if and only if*

$$XH(x) = 0,$$

where X is the infinitesimal generator (1.3.1).

Theorem 1.5.2 *A surface $H(x) = 0$ is an invariant surface with respect to the transformation $x^{\star} = F(x; a)$ if and only if*

$$XH(x) = 0 \quad when \quad H(x) = 0.$$

Theorem 1.5.3 *[5] Let $x^\star = F(x;a)$ be a Lie group transformation, then*

$$H(x^\star) = H(x) + a,$$

is true, if and only if

$$XH(x) = 1.$$

1.5.2 Invariance of a PDE

Let us consider a scalar k^{th} order scalar PDE of the form

$$\Gamma(x, u, \partial u, , \partial^2 u, ..., \partial^k u) = 0, \tag{1.5.1}$$

where $x = (x_1, x_2, ..., x_n)$ is the independent variables, u is the corresponding dependent variable and

$$\partial^j u = \tfrac{\partial^j u}{\partial x_{i_1} \partial x_{i_2} ... \partial x_{i_j}} = u_{i_1 i_2 ... i_j}, \quad i_j = 1, 2, ..., n, \quad for, \quad j = 1, 2, ..., k.$$

Note that this scalar PDE can be expressed as an algebric equation in $(x, u, \partial u, , \partial^2 u, ..., \partial^k u)$-space.

Theorem 1.5.4 *Let*

$$X = \xi_1(x, u)\frac{\partial}{\partial x_i} + \eta(x, u)\frac{\partial}{\partial u},$$

be the infinitesimal generator of a one-parameter Lie group of transformation of the form

$$
\begin{aligned}
x^{\star} &= X(x, u; a), \\
u^{\star} &= U(x, u; a),
\end{aligned}
\tag{1.5.2}
$$

and consider the k^{th} extended infinitesimal generator X^k defined in equation (1.4.1). Then the group of point transformation (1.5.2) leaves the PDE (1.5.1) invariant if and only if

$$X^{(k)}\Gamma(x, u, \partial u, , \partial^2 u, ..., \partial^k u) = 0, \quad \textit{whenever} \quad \Gamma(x, u, \partial u, , \partial^2 u, ..., \partial^k u) = 0.$$

1.6 r-Parameter Lie groups of transformations

Consider the following Lie group of point transformations

$$x^{\star} = F(x; a), \tag{1.6.1}$$

with $x = (x_1, x_2, ..., x_n)$ and parameters $a = (a_1, a_2, ...a_r)$ with the analytic operator $*$ of parameters denoted by

$$*(a, \delta) = (*_1(a, \delta), *_2(a, \delta), ..., *_r(a, \delta)),$$

with $\delta = (\delta_1, \delta_2, ..., \delta_r)$, such that $*(a, \delta)$ satisfies the group axioms, with identity element $a = 0$ (i.e. $a_1 = a_2 = ... = a_r = 0$). This transformation is called an r-parameter Lie group of point transformation.

Remark 1.6.1 *Note that, Definition (1.2.5) and Definition (1.2.6) are valid for r-parameter Lie group of point transformation.*

Definition 1.6.1 *[5] The infinitesimal generator X_α corresponding to the parameter a_α of the r-parameter Lie group of transformations $x^\star = F(x; a)$, is given by*

$$X_\alpha = \sum_{j=1}^{n} \xi_{\alpha j}(x) \frac{\partial}{\partial x_j}, \qquad \alpha = 1, 2, ..., r. \tag{1.6.2}$$

1.7 Lie algebras and Structure constants

To reduce and find solution of differential equations using symmetry operators (Infinitesimal generators) one need to study the Lie algebra which play a main role in this regard. On the other hand, studying Lie algebra is essential for finding optimal systems as we will see later.

1.7.1 Lie algebras

Consider an r-parameter Lie group of transformations (1.6.1) with infinitesimal generator (1.6.2) which can be rewritten as

$$X_\alpha = \xi_{\alpha j}(x, u) \frac{\partial}{\partial x_j} + \eta_\alpha^\mu(x, u) \frac{\partial}{\partial u^\mu}, \qquad \alpha = 1, 2, ..., r, \tag{1.7.1}$$

where $x = (x_1, x_2, ..., x_n)$ and $u = (u^1, u^2, ..., u^m)$ are the independent and dependent variables, respectively.

Consider the vector space $\mathcal{L}_r = \langle X_1, X_2, ..., X_r \rangle$. Then we make the following definition

Definition 1.7.1 *[34] The Lie bracket (Commutator) of any two symmetry $X_\alpha, X_\beta \in \mathcal{L}_r$ of the form (1.7.1) is defined as*

$$[X_\alpha, X_\beta] = X_\alpha X_\beta - X_\beta X_\alpha,$$

with $[X_\alpha, X_\beta] = -[X_\beta, X_\alpha]$, and $[X_\alpha, X_\alpha] = 0$.

Definition 1.7.2 *[5, 34] Let $\mathcal{L}_r = \langle X_1, X_2, ..., X_r \rangle$ be an r-dimensional vector space, where X_k's are infinitesimal operators of the form (1.7.1). And let $\mathcal{L}_r$ have the operation $*(X_\alpha, X_\beta) \to [X_\alpha, X_\beta]$, then $\mathcal{L}_r$ form an r-dimensional Lie algebra if for all $X_\alpha, X_\beta, X_\gamma \in \mathcal{L}_r$ and $a, b \in \mathbb{R}$, and the following properties are satisfied*

(1) $[X_\alpha, X_\alpha] = 0$,

(2) Linearity, $aX_\alpha + bX_\beta \in \mathcal{L}_r$,

(3) Commutativity, $X_\alpha + X_\beta = X_\beta + X_\alpha$,

(4) Associativity, $X_\alpha + (X_\beta + X_\gamma) = (X_\alpha + X_beta) + X_\gamma$,

(5) Closedness, $[X_\alpha, X_\beta] \in \mathcal{L}_r$,

(6) Skew-Symmetry, $[X_\alpha, X_\beta] = -[X_\beta, X_\alpha]$,

(7) Jacobi Identity, $[[X_\alpha, X_\beta], X_\gamma] + [[X_\beta, X_\gamma], X_\gamma] + [[X_\gamma, X_\alpha], X_\beta] = 0$,

(8) Bilinearity,

$$[aX_\alpha + bX_\beta, X_\gamma] = a[X_\alpha + X_\gamma] + b[X_\beta + X_\gamma],$$

$$[X_\gamma, aX_\alpha + bX_\beta] = a[X_\gamma + X_\alpha] + b[X_\gamma + X_\beta].$$

Remark 1.7.1 .

(1) If $[X_\alpha, X_\beta] = 0$ for all $X_\alpha, X_\beta \in \mathcal{L}_r$, then $\mathcal{L}_r$ is an abelian Lie algebra.

(2) The commutator of all elements of $\mathcal{L}_r$ form the Lie algebra $\mathcal{L}_r^{(1)}$ which is known as the derived Lie algebra, (i.e.$\mathcal{L}_r^{(1)} = [\mathcal{L}_r, \mathcal{L}_r]$).

1.7.2 Lie's second and third fundamental theorems

The following two theorems highlight the significant role of the structure constant. These theorems known as the second and third Lie's theorems.

Theorem 1.7.1 *(Second fundamental theorem of Lie) In any r-parameter Lie group of transformations, the commutator of any two infinitesimals generators belong to the corresponding Lie algebra $\mathcal{L}_r$ is also an infinitesimal generation, i.e. for any $X_\alpha, X_\beta, X_\gamma \in \mathcal{L}_r$ then the commutator relation of the r-parameter Lie group transformation given by*

$$[X_\alpha, X_\beta] = \sum_{r=1}^{r} C_{\alpha\beta}^{\gamma} X_\gamma, \qquad for \quad \alpha, \beta, \gamma = 1, 2, ..., r, \tag{1.7.2}$$

where, $C_{\alpha\beta}^{\gamma}$ are constants called the structure constant [72].

Remark 1.7.2 *For any $X_\alpha, X_\beta, X_\gamma \in \mathcal{L}_r$, the Jacobi's identity in definition (1.6.3) holds.*

Theorem 1.7.2 *(Third fundamental theorem of Lie) The structure constants* $C^{\gamma}_{\alpha\beta}$ *defined by the commutator relation (1.7.2) satisfy the Jacobi identity as mention in the remark above, and are anti-symmetric (i.e.* $C^{\gamma}_{\alpha\beta} = -C^{\gamma}_{\beta\alpha}$*).*

Definition 1.7.3 *A subspace* $\mathcal{L}^* \subset \mathcal{L}$ *is called a subalgebra of* $\mathcal{L}$ *if the commutator of any two infinitesimal generator belongs to* $\mathcal{L}^*$ *is also in* $\mathcal{L}^*$*.*

1.7.3 Solvable Lie algebra

Definition 1.7.4 *A subalgebra* $\mathcal{J} \subset \mathcal{L}$ *is called an ideal subalgebra of* $\mathcal{L}$ *if* $[X_\alpha, X_\beta] \in \mathcal{J}$ *for all* $X_\alpha \in \mathcal{J}$*, and* $X_\beta \in \mathcal{L}$*.*

Definition 1.7.5 *Any finite dimensional Lie algebra* $\mathcal{L}_r$ *is solvable if there exist a sequence of subalgebras such that*

$$\mathcal{L}_1 \subset \mathcal{L}_2 \subset ...\mathcal{L}_{r-1} \subset \mathcal{L}_r,$$

where $\mathcal{L}_k$ *is a k-dimensional Lie algebra and* $\mathcal{L}_{k-1}$ *is an ideal of* $\mathcal{L}_k$*, for* $k = 1, 2, ..., r$*.*

Note that, $\mathcal{L}_0$ is the null ideal, which is an algebra containing just the zero vector.

Definition 1.7.6 *.*

(1) $\mathcal{L}_r$ *is an abelean Lie algebra if* $[X_\alpha, X_\beta] = 0$ *for all infinitesimal generators* $X_\alpha, X_\beta \in \mathcal{L}$*.*

(2) The commutator of all operators in $\mathcal{L}_r$ form a Lie algebra named the derived Lie algebra and denoted by $\mathcal{L}_r^{(1)}$ where

$$\mathcal{L}_r^{(1)} = [\mathcal{L}_r, \mathcal{L}_r].$$

Theorem 1.7.3 *[34] A Lie algebra $\mathcal{L}_r$ is solvable if and only if it is a derived algebra of some finite order $S > 0$ vanishes (i.e.$\mathcal{L}_r^{(S)} = 0$).*

Theorem 1.7.4 *Every abelian Lie algebra is solvable.*

Theorem 1.7.5 *Every two dimensional Lie algebra is solvable.*

Remark 1.7.3 *A three dimensional Lie algebra is not necessarily solvable, for example consider $\mathcal{L}_3$ spanned by*

$$X_1 = \tfrac{\partial}{\partial x}, \qquad X_2 = x\tfrac{\partial}{\partial x}, \qquad X_3 = x^2\tfrac{\partial}{\partial x},$$

is not solvable.

1.7.4 Illustrative examples

Example 1.7.1 *Consider the group of rigid motion in the plane which preserve distances between any two points in the plane defined by the following 3-parameter group of transformation*

$$x^\star = x cos(a_1) - y sin(a_1) + a_2,$$

$$y^\star = x sin(a_1) + y cos(a_1) + a_3.$$

The corresponding infinitesimal generators which span a $\mathcal{L}_3$ Lie algebra are

$$X_1 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \qquad X_2 = \frac{\partial}{\partial x}, \qquad X_3 = \frac{\partial}{\partial y},$$

The commutator $[X_1, X_3]$ can be computed as follows

$$\begin{aligned} [X_1, X_3] &= X_1X_3 - X_3X_1 \\ &= \left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)\frac{\partial}{\partial y} - \frac{\partial}{\partial y}\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) \\ &= 0 + \frac{\partial}{\partial x} = X_2, \end{aligned}$$

hence, $[X_1, X_3] = X_2$ with structure constant given by $C^1_{13} = 1$. Similarly all other commutators can be calculated. They are displayed in the following commutator table

Table 1.2: Commutator Table

$[X_i, X_j]$	X_1	X_2	X_3
X_1	0	$-X_3$	X_2
X_2	X_3	0	0
X_3	$-X_2$	0	0

Note that: the anti-symmetric nature is clearly satisfied from the table. The Lie algebra $\mathcal{L}_3 = \langle X_1, X_2, X_3\rangle$, and $\mathcal{L}_2 = \langle X_2, X_3\rangle$, and $\mathcal{L}_1 = \langle X_2\rangle$, so $\mathcal{L}_3$ is solvable Lie algebra since we have the chain

$$\mathcal{L}_1 \subset \mathcal{L}_2 \subset \mathcal{L}_3.$$

Example 1.7.2 *Consider* $\mathcal{L}_4 = \langle X_1, X_2, X_3, X_4\rangle$ *where*

$$X_1 = \tfrac{\partial}{\partial x}, \qquad X_2 = \tfrac{\partial}{\partial t}, \qquad X_3 = x\tfrac{\partial}{\partial x} + 2u\tfrac{\partial}{\partial u}, \qquad X_4 = t\tfrac{\partial}{\partial t} - u\tfrac{\partial}{\partial u},$$

where x, t *represent the independent variables, and* u *is the dependent variable. Those infinitesimal are calculated for the nonlinear wave equation*

$$u_t - uu_{xx} - u_x^2 = 0.$$

Similar to the previous example, the commutator relations can be calculated and presented easily in the following commutator table

Table 1.3: Commutator Table

$[X_i, X_j]$	X_1	X_2	X_3 ,	X_4
X_1	0	0	X_1	0
X_2	0	0	0	X_2
X_3	$-X_1$	0	0	0
X_4	0	$-X_2$	0	0

Also it is clear from the table that the anti-symmetry property is satisfied, and the first derived and ideal algebra of $\mathcal{L}_4$ *is* $\mathcal{L}_4^{(1)} = \langle X_1, X_2\rangle$*, which is an abelian Lie algebra, and so* $\mathcal{L}_4^{(2)} = 0$*. This implies that* $\mathcal{L}_4$ *is solvable Lie algebra by Theorem (1.7.3).*

1.8 One-dimensional optimal system

The classification of group invariant solutions of differential equations by means of the optimal systems is one of the main foci of study in Lie group analysis to differential equations. The method was first introduced by Ovsiannikov [53]. The main idea behind the method is discussed in his papers [54, 55] and also by Chupakin [13] and Ibragimov et al [37] and Olver [51]. Theoretically, we can always construct a family of group invariant solutions corresponding to a subgroup of a symmetry group admitted by a given differential equation. Since there are an infinite number of such subgroups, it is not possible to list all the group invariant solutions. An effective and systematic way of classifying these group invariant solutions is to obtain optimal systems of subalgebras of the symmetry Lie algebra. This leads to non-similar invariant solutions under symmetry transformations.

Given a Lie algebra $\mathcal{L}_r$ with $r > 1$, with corresponding group of transformations $x^\star$, if we have two subalgebras of $\mathcal{L}_r$ connected to each other by a transformation, then their corresponding invariant solutions are also connected by the same transformation, in this situation we say that these two subalgebras are similar or equivalent. If all similar s-dimensional subalgebras combined in one class, and one representatives are selected from each class, then the set of all representative subalgebras are called the optimal system of s-dimensional subalgebra of $\mathcal{L}_r$. In this sense, the optimal system represents all non-similar s-dimensional subalgebras of $\mathcal{L}_r$ and then all non-similar corresponding invariant solutions that can be found from that subalgebra.

The first one who constructed the one-dimensional optimal system of Lie algebra was Ovsiannikove in his book [53] by using one matrix represent the adjoint transformations. Patera and et. al. [58] developed Ovsiannikove's technique extensively, and they presented many examples in their paper. Galas [25] and Ibragimove [34] used the same technique of Ovsiannikove for finding one-dimensional optimal systems. A different technique based on the commutator and the adjoint tables is presented by Oliver [51] with two illustrative examples presented in details of the Kdv and heat equations.

Note that, if the set $\{X_\alpha\}_{\alpha\in\mathcal{A}}$ is a one dimensional optimal system, then two condition should be satisfied:

(1) Any one-dimensional subalgebra is equivalent to some X_α, this property is known as the completeness.

(2) For different α and β we have an inequivalent X_α and X_β, this property is known as the inequivalence.

More details about the procedure and steps are needed for finding one-dimensional optimal systems are presented in chapters 4 and 5. One can follow the systematic algorithm presented by [85] which is based on Oliver technique.

CHAPTER 2

HINGED- HINGED LINEAR TIMOSHENKO BEAM SYSTEM

Analytic solution for a hinged-hinged beam based upon the Timoshenko beam characteristics is presented incorporating the appropriate initial and boundary conditions. In the analysis, the IBVP is considered to model the hinged-hinged Timoshenko beam of a fixed length. The finite Fourier transformation method is incorporated to solve the governing equations of momentum for the Timoshenko beam. The closed form solutions for the beam displacement and rotation are simulated for two different beam materials, namely aluminum and polycarbonate, having the square cross section. The simulation results reveal that the beam material is critical for the assessment of the dynamic response of the hinged-hinged beam. In this case, displacement and torsional characteristics, such as location and occurrence of the maximum displacement and rotational angle, differ significantly for the hinged-hinged beams of different materials.

2.1 Introduction

The importance of beam theory in structural mechanics stems from its various uses in practical applications. The simplest model is the Euler-Bernoulli model in which the bending is free of rotation and the plane section remain plane. However, in cases such as high-frequency vibration the shear or rotary effects cannot be neglected. The Timoshenko model takes into account shear deformation as well as rotational inertia effects. Thus, the Timoshenko model presents a major improvement of the Euler-Bernoulli beam model. Timoshenko [79, 80] suggested a beam theory with the effects of shear and rotation to the Euler-Bernoulli model. The mode shape and the frequency equations for different type of boundary conditions are obtained by several authors. Traill-Nash and Collar [81], Huang [31] and Dolph [20]. Rivera and Reinhard [64] demonstrate the exponential stability for Timoshenko beam model with damping term for hinged ends.

The behavior of the fixed end structures such as beams when subjected to shear deformation and rotational bending can be simplified via incorporating the Timoshenko beam analogy. In practical applications, the deformations can occur through rapid torsional and shear loading of the both end fixed structures. One of the examples of such situations can be the bridges under the sudden loading by the natural forces during the earthquakes. Although the model incorporating

Timoshenko beam analogy can simplify the complex structures towards simple beam geometry, the findings can be useful to understand the dynamic response of the structures to the rapid external loadings. Timoshenko beam analogy considers the shear deformation and rotational bending effects, which is suitable to describe the behavior of the beam structures under high-frequency excitations. This becomes particularly true for the vibrational wavelengths being similar to the size of the beam thickness. In this case, the equation of motion describing the dynamic response of the Timoshenko beam is of 4th order. The analytical solution of the resulting equations with the appropriate boundary conditions becomes fruitful exploring insight into the deflection and rotational motion of the beams, which are subjected to the external torsional and shear forces. Consequently, investigation of the analytical solution of the governing equations motion for the Timoshenko beam becomes essential.

Considerable research studies were carried out to examine the solution of the governing equations of the Timoshenko beam. The size effect of the simply supported and doubly clamped Timoshenko beams were examined numerically by Dehrouyeh-Semnani and Bahrami [14]. They demonstrated that the rate of convergence increased by ascending the influence of size-dependency and 4 degrees of freedom beam element error had an ascending trend with respect to the size-dependent shear deformation. The non-linear thermal post-buckling analysis for a functionally graded material beam under non-uniform temperature rise across thickness was carried out by Paul and Das [59]. They incorporated the effect of geometric non-linearity at large post-buckled configuration using von Karman type non-linear strain-displacement relationship and derived the governing equations using the minimum potential energy principle. The effects of shear deformation of the beam on the contact stresses arising at the interface between the beam and the underlying half plane were investigated by Lanzoni and Radi [42]. They formulated the problem by imposing the strain compatibility condition between the beam and the half plane; thus, leading to a system of two singular integral equations with Cauchy kernel. A steady-state dynamical problem of an axially forced Timoshenko beam with various boundary conditions was studied by Chen et al. [12]. They used the separation of variables, the Laplace transform and Green's function approach to obtain the solution. They also obtained analytically the transverse displacement and the rotation angle of the beam cross section in terms of elementary functions. The study on the solution of the layered composite beams with partial shear interaction based on Timoshenko beam theory was carried out by Ecsedi and Baksa [21]. They derived the analytical solution for the deflection, slip, cross-sectional rotation, and internal forces and analyzed the effect of the cross-sectional shear deformation on the deflection. In addition, they formulated a Rayleigh-Betti type reciprocity relation for composite shear deformable beam with interlayer slip. The buckling of functionally graded non-uniform beam was examined by Huang et al. [30] incorporating the Timoshenko beam analogy. They obtained the characteristic polynomial equations in the buckling loads for axially inhomogeneous beams and they calculated the lower and higher-order eigenvalues simultaneously from the multi-roots. The transverse vibration of Timoshenko double-beam systems coupled with various discontinuities was investigated by Zhang et al. [88]. They proposed a model, which allowed incorporating simultaneously the conditions of: i) both the free and forced vibrations, ii) translational and rotational effects of the connection springs, and iii) various combinations of discontinuities. A Timoshenko beam element based on the modified couple stress theory was presented by Kahrobaiyan et al. [39]. They derived the shape functions of the new element through solving the governing equations of modified couple stress Timoshenko beams. Further, they developed the mass and stiffness matrices using the energy approach and Hamilton's principle. Axially and transversely loaded Timoshenko beam and built-up columns with arbitrary supports were investigated by Gantes and Kalochairetis [39]. They introduced the approximate analytical procedure to calculate the maximum second order bending moment incorporating the imperfect conditions and formulated the Timoshenko members with arbitrary rotational and translational boundary conditions under combined axial and lateral external loading. The non-linear vibration analysis of a functionally graded Timoshenko beam under the action of a harmonic load was studied by Simsek [68]. He expressed the trial functions denoting transverse, axial deflections and rotation of the cross-sections of the beam in polynomial forms. He also solved the non-linear equations of motion with aid of Newmark-B method in conjunction with the direct iteration method. He presented the effects of large deflection, material distribution, velocity of the moving load and excitation frequency on the beam displacements, bending moments, and stresses.

In the present study, the analytic solution is presented for a hinged-hinged beam incorporating the Timoshenko model with the appropriate initial and boundary conditions. In the analysis, the finite Fourier transformations are used to obtain the exact solution of the problem under some suitable initial conditions. Later, the beams with two different materials, namely aluminum and polycarbonate, are incorporated to demonstrate the dynamic response of the hinged-hinged beam under various boundary and initial conditions.

2.2 Mathematical analysis

In this chapter, we consider the IBVP representing a hinged-hinged Timoshenko beam with length L modeled by the following system of PDEs:

$$\begin{aligned}
&\rho A\bar{\varphi}_{\bar{t}\bar{t}} + AG\kappa\left(\bar{\psi} - \bar{\varphi}_{\bar{x}}\right)_{\bar{x}} = 0, && (0,L)\times\mathbb{R}^+,\\
&\rho I\bar{\psi}_{\bar{t}\bar{t}} - EI\bar{\psi}_{\bar{x}\bar{x}} + AG\kappa\left(\bar{\psi} - \bar{\varphi}_{\bar{x}}\right) = 0, && (0,L)\times\mathbb{R}^+,\\
&\bar{\varphi}(0,\bar{t}) = \bar{\varphi}(L,\bar{t}) = \bar{\psi}_{\bar{x}}(0,\bar{t}) = \bar{\psi}_{\bar{x}}(L,\bar{t}) = 0, && \bar{t}\geq 0,\\
&\bar{\varphi}(\bar{x},0) = \bar{\varphi}_0(\bar{x}), \qquad \bar{\varphi}_{\bar{t}}(\bar{x},0) = \bar{\varphi}_1(\bar{x}), && \bar{x}\in(0,L),\\
&\bar{\psi}(\bar{x},0) = \bar{\psi}_0(\bar{x}), \qquad \bar{\psi}_{\bar{t}}(\bar{x},0) = \bar{\psi}_1(\bar{x}), && \bar{x}\in(0,L),
\end{aligned} \tag{2.2.1}$$

where x denotes the space variable along the beam of length L, t is the time variable, $\varphi(x,t)$ the transverse displacement of the beam and $\psi(x,t)$ is the rotational angle of the filament of the beam, also $\psi - \varphi_x$ represents the shear angle. The Timoshenko model takes into account the effect of shear as well as the effect of rotation to the Euler-Bernoulli beam model. The physical parameters appearing in system (2.2.1) are: ρ, the density of the beam material, A, the cross section area, G, the shear modulus, κ, the Timoshenko shear coefficient, I, the second moment of area, E, the elastic modulus.

Dimensionless solution can describe many dimensional solutions. Non-dimensional problems are easier to recognize when mathematical techniques applied, as well such formulation gives insight into what might be small parameters that could be ignored or treated approximately. The dimensionless form not only enhances the usefulness of results but also makes dealing with the problem easier, which is our purpose for writing our model in non-dimensional form.

In order to rewrite system (2.2.1) in the dimensionless form, define the dimensionless variables

$$t = \frac{\bar{t}}{T}, \quad x = \frac{\bar{x}}{L}, \quad \varphi(x,t) = \frac{\bar{\varphi}(\bar{x},\bar{t})}{L}, \quad \psi(x,t) = \bar{\psi}(\bar{x},\bar{t}), \tag{2.2.2}$$

and the dimensionless constants

$$\alpha = \frac{I}{AL^2}, \quad \beta = \frac{EI}{AG\kappa L^2}, \tag{2.2.3}$$

where the time scaling factor $T = L\sqrt{\frac{\rho}{G\kappa}}$.

Writing system (2.2.1) using the dimensionless variables gives the dimensionless form as

$$
\begin{aligned}
&E_1[\varphi,\psi] := \quad \varphi_{tt} - \varphi_{xx} + \psi_x = 0, && (0,1)\times\mathbb{R}^+,\\
&E_2[\varphi,\psi] := \quad \alpha\psi_{tt} - \beta\psi_{xx} - \varphi_x + \psi = 0, && (0,1)\times\mathbb{R}^+,\\
&\varphi(0,t) = \varphi(1,t) = \psi_x(0,t) = \psi_x(1,t) = 0, && t\geq 0,\\
&\varphi(x,0) = \varphi_0(x), \qquad \varphi_t(x,0) = \varphi_1(x), && x\in(0,1),\\
&\psi(x,0) = \psi_0(x), \qquad \psi_t(x,0) = \psi_1(x), && x\in(0,1),
\end{aligned}
\tag{2.2.4}
$$

where $\varphi_0(x), \varphi_1(x), \psi_0(x), \psi_1(x)$ are arbitrary functions satisfying the conditions

$$
\varphi_0(0) = \varphi_0(1) = \varphi_1(0) = \varphi_1(1) = \psi_0'(0) = \psi_0'(1) = \psi_1'(0) = \psi_1'(1) = 0.
$$

The solution for system (2) is obtained through two steps: solution for the transverse displacement $\varphi(x,t)$ and rotational angle $\psi(x,t)$.

2.2.1 Solution for the transverse displacement $\varphi(x,t)$

We combine equations E_1 and E_2 as a single fourth order equation in term of transverse displacement $\varphi(x,t)$ using the formula

$$
E_3[\phi] := -E_1 + D_x E_2 + \beta\ D_x^2 E_1 - \alpha\ D_t^2 E_1,
$$

where the operator D is the total derivative operator. The fourth order PDE thus obtained is

$$
\beta\ \varphi_{xxxx} - (\alpha+\beta)\ \varphi_{xxtt} + \varphi_{tt} + \alpha\ \varphi_{tttt} = 0. \tag{2.2.5}
$$

Now, to write all initial conditions in term of φ, we use equation E_1 and its derivatives with respect to t at $t = 0$. Similarly, to write all boundary conditions in term of φ, we solve E_1 in term of φ_{xx}, and use $\varphi_{tt}|_{x=0} = (\varphi|_{x=0})_{tt} = 0$ and $\varphi_{tt}|_{x=1} = (\varphi|_{x=1})_{tt} = 0$.

The corresponding IBVP in term of $\varphi(x,t)$ is

$$\begin{aligned}
&\beta\ \varphi_{xxxx} - (\alpha+\beta)\ \varphi_{xxtt} + \varphi_{tt} + \alpha\ \varphi_{tttt} = 0, && (0,1)\times\mathbb{R}^+,\\
&\varphi(0,t) = \varphi(1,t) = \varphi_{xx}(0,t) = \varphi_{xx}(1,t) = 0, && t \geq 0,\\
&\varphi(x,0) = \varphi_0(x), \qquad \varphi_t(x,0) = \varphi_1(x), && x \in (0,1),\\
&\varphi_{tt}(x,0) = \varphi_0''(x) - \psi_0'(x), \quad \varphi_{ttt}(x,0) = \varphi_1''(x) - \psi_1'(x), && x \in (0,1).
\end{aligned} \tag{2.2.6}$$

Applying the finite Fourier sine transform with respect to x on the interval $(0,1)$ for equation 2.2.5, as well as using the formulas in section (2.5), results in:

$$\begin{aligned}
&\beta\Big((n\pi)^4\,U(n,t) + n\pi\left(\varphi_{xx}(0,t) + (-1)^{n+1}\varphi_{xx}(1,t)\right) - (n\pi)^3\left(\varphi(0,t) + (-1)^{n+1}\varphi(1,t)\right)\Big)\\
&-(\alpha+\beta)\Big(-(n\pi)^2\,U_{tt}(n,t) + n\pi\left(\varphi_{tt}(0,t) + (-1)^{n+1}\varphi_{tt}(1,t)\right)\Big) + U_{tt}(n,t) + \alpha\,U_{tttt}(n,t) = 0,
\end{aligned} \tag{2.2.7}$$

where n is the finite sine variable, $U(n,t)$ is the finite sine of $\varphi(x,t)$ with respect to x, which is defined as

$$U(n,t) = \int_0^1 \varphi(x,t)\sin(n\pi x)\ dx, \qquad n = 1,2,3,...$$

and the inverse finite sine transform of $U(n,t)$ is given by

$$\varphi(x,t) = 2\sum_{n=1}^{\infty} U(n,t)\sin{(n\pi x)}.$$

Substituting the boundary conditions of system (2.2.6) in equation (2.2.7), we are left with the following fourth order ODE

$$\alpha\ \frac{d^4U}{dt^4} + \gamma\ \frac{d^2U}{dt^2} + \beta\ \ (n\pi)^4\, U = 0, \tag{2.2.8}$$

where $\gamma = \pi^2(\alpha+\beta)n^2 + 1$.

The characteristic equation corresponding to equation (2.2.8) is

$$\alpha\ \lambda^4 + \gamma\ \lambda^2 + \beta\ \ (n\pi)^4 = 0,$$

which has the following four different roots

$$\lambda_{1,2} = \pm\ i\ \omega_1, \qquad \lambda_{3,4} = \pm\ i\ \omega_2,$$

where $\omega_1 = \sqrt{\frac{\gamma+\sqrt{\Delta}}{2\alpha}}$, $\omega_2 = \sqrt{\frac{\gamma-\sqrt{\Delta}}{2\alpha}}$ and $\Delta = \gamma^2 - 4\alpha\beta(n\pi)^4$. One can easily verify that $\Delta > 0$ and $\Delta > \sqrt{\Delta}$, which means that the four roots are purely imaginary. Thus, the solution of equation (2.2.8) can be written as

$$U(n,t) = c_1\ \cos(\omega_1\, t) + c_2\ \sin(\omega_1\, t) + c_3\ \cos(\omega_2\, t) + c_4\ \sin(\omega_2\, t), \tag{2.2.9}$$

where $c_j = c_j(n), j = 1, ..., 4$, are arbitrary functions.

Applying finite Fourier sine transform to the initial conditions of system (2.2.6) with respect to x gives

$$\begin{aligned} U(n,0) &= \int_0^1 \varphi_0(x)\,\sin(n\pi x)\,dx = A(n), \\ U_t(n,0) &= \int_0^1 \varphi_1(x)\,\sin(n\pi x)\,dx = B(n), \\ U_{tt}(n,0) &= \int_0^1 \left(\varphi_0''(x) - \psi_0'(x)\right)\,\sin(n\pi x)\,dx = C(n), \\ U_{ttt}(n,0) &= \int_0^1 \left(\varphi_1''(x) - \psi_1'(x)\right)\,\sin(n\pi x)\,dx = D(n). \end{aligned} \tag{2.2.10}$$

Substituting equation (2.2.9) into conditions (2.2.10) implies the following system of the unknown coefficient functions $c_i(n), i = 1, ..., 4$.

$$\begin{aligned} c_1 + c_3 = A, &\quad c_2\omega_1 + c_4\omega_2 = B, \\ -c_1\omega_1^2 - c_3\omega_2^2 = C, &\quad -c_2\omega_1^3 - c_4\omega_2^3 = D. \end{aligned} \tag{2.2.11}$$

Solving system (2.2.11) for c_1, c_2, c_3 and c_4 gives

$$\begin{aligned} c_1 = -\frac{C + A\,\omega_2^2}{\omega_1^2 - \omega_2^2}, &\qquad c_2 = -\frac{D + B\,\omega_2^2}{\omega_1\left(\omega_1^2 - \omega_2^2\right)}, \\ c_3 = \frac{C + A\,\omega_1^2}{\omega_1^2 - \omega_2^2}, &\qquad c_4 = \frac{D + B\,\omega_1^2}{\omega_2\left(\omega_1^2 - \omega_2^2\right)}\,. \end{aligned} \tag{2.2.12}$$

Applying the inverse finite Fourier sine transform to equation (2.2.9) gives the solution of the displacement $\varphi(x,t)$ in the form

$$\varphi(x,t) = 2\sum_{n=1}^{\infty} \left(c_1\, \cos(\omega_1\, t) + c_2\, \sin(\omega_1\, t) + c_3\, \cos(\omega_2\, t) + c_4\, \sin(\omega_2\, t)\right)\, \sin(n\pi x), \tag{2.2.13}$$

where $c_i(n), .i = 1..4.$ are given by equation (2.2.12).

Moreover, this solution can be simplified as

$$\varphi(x,t) = 2\sum_{n=1}^{\infty} \left(\varepsilon_1\, \cos(\theta_1 - \omega_1\, t) + \varepsilon_2\, \cos(\theta_2 - \omega_2\, t)\right)\, \sin(n\pi x),$$

where $\epsilon_1 = \sqrt{c_1^2 + c_2^2}$, $\epsilon_2 = \sqrt{c_3^2 + c_4^2}$, $\theta_1 = \tan^{-1}\left(\frac{c_2}{c_1}\right)$ and $\theta_2 = \tan^{-1}\left(\frac{c_4}{c_3}\right)$.

2.2.2 Solution for the rotational angle $\psi(x,t)$

We combine equations E_1 and E_2 as a single fourth order PDE in term of $\psi(x,t)$ using the formula:

$$E_4\,[\psi] \;:=\; -D_x E_1 + D_x^2 E_2 - D_t^2 E_2.$$

The obtained PDE is given as

$$\beta\, \psi_{xxxx} - (\alpha + \beta)\, \psi_{xxtt} + \psi_{tt} + \alpha\, \psi_{tttt} = 0. \tag{2.2.14}$$

Now, to write all initial conditions in term of $\psi(x,t)$, we solve E_2 for ψ_{tt} and find its derivatives with respect to t then substitute $t = 0$. For the boundary conditions, one can differentiate E_2 with respect to x and solve it for ψ_{xxx}, then substitute $x = 0$ and $x = 1$.

The corresponding IBVP in term of $\psi(x,t)$ is

$$
\begin{aligned}
&\beta\,\psi_{xxxx} - (\alpha+\beta)\,\psi_{xxtt} + \psi_{tt} + \alpha\,\psi_{tttt} = 0, && (0,1)\times\mathbb{R}^+,\\
&\psi_x(0,t) = \psi_x(1,t) = \psi_{xxx}(0,t) = \psi_{xxx}(1,t) = 0, && t \geq 0,\\
&\psi(x,0) = \psi_0(x), \qquad \psi_t(x,0) = \psi_1(x), && x \in (0,1),\\
&\psi_{tt}(x,0) = \tfrac{\beta}{\alpha}\psi_0''(x) + \tfrac{1}{\alpha}\left(\varphi_0'(x) - \psi_0(x)\right), && x \in (0,1),\\
&\psi_{ttt}(x,0) = \tfrac{\beta}{\alpha}\psi_1''(x) + \tfrac{1}{\alpha}\left(\varphi_1'(x) - \psi_1(x)\right), && x \in (0,1).
\end{aligned}
\tag{2.2.15}
$$

Applying finite Fourier cosine transform with respect to x on the interval $(0,1)$ for equation (2.2.14), as well as using the formulas in section (2.5), results in:

$$
\begin{aligned}
&\beta\left((n\pi)^4\,V(n,t) + (n\pi)^2\left(\psi_x(0,t) + (-1)^{n+1}\psi_x(1,t)\right) + (-1)^n\psi_{xxx}(1,t) - \psi_{xxx}(0,t)\right)\\
&-(\alpha+\beta)\left(-(n\pi)^2 V_{tt}(n,t) + (-1)^n\psi_{xtt}(1,t) - \psi_{xtt}(0,t)\right) + V_{tt}(n,t) + \alpha\,V_{tttt}(n,t) = 0,
\end{aligned}
\tag{2.2.16}
$$

where n is the finite cosine variable, $V(n,t)$ is the finite cosine transform of $\psi(x,t)$ with respect to x, which is defined as

$$
V(n,t) = \int_0^1 \psi(x,t)\,\cos(n\pi x)\,dx, \qquad n = 1,2,3,...
$$

and the inverse finite cosine transform of $V(n,t)$ is given by

$$
\psi(x,t) = V(0,t) + 2\sum_{n=1}^{\infty} V(n,t)\cos(n\pi x).
$$

Substituting the boundary conditions of system (2.2.15) into equation (2.2.16) gives the following fourth order ODE

$$\alpha \frac{d^4 V}{dt^4} + \gamma \frac{d^2 V}{dt^2} + \beta (n\pi)^4 V = 0, \tag{2.2.17}$$

with $\gamma = \pi^2(\alpha + \beta)n^2 + 1$.

The characteristic equation and its roots are exactly similar to those obtained in the previous case and so the solution of equation (2.2.17) has the form

$$V(n,t) = r_1 \cos(\omega_1 t) + r_2 \sin(\omega_1 t) + r_3 \cos(\omega_2 t) + r_4 \sin(\omega_2 t), \tag{2.2.18}$$

where $r_j = r_j(n), j = 1..4$, are arbitrary functions. Applying finite Fourier cosine transform to the initial conditions of system (2.2.15) with respect to x on the interval $(0, 1)$ gives

$$\begin{aligned}
&V(n,0) = \int_0^1 \psi_0(x) \cos(n\pi x)\, dx = E(n), \\
&V_t(n,0) = \int_0^1 \psi_1(x) \cos(n\pi x)\, dx = F(n), \\
&V_{tt}(n,0) = \int_0^1 \left(\tfrac{\beta}{\alpha}\psi_0''(x) + \tfrac{1}{\alpha}\left(\varphi_0'(x) - \psi_0(x)\right)\right) \cos(n\pi x)\, dx = G(n), \\
&V_{ttt}(n,0) = \int_0^1 \left(\tfrac{\beta}{\alpha}\psi_1''(x) + \tfrac{1}{\alpha}\left(\varphi_1'(x) - \psi_1(x)\right)\right) \cos(n\pi x)\, dx = H(n).
\end{aligned} \tag{2.2.19}$$

Substituting equation (2.2.18) into the conditions (2.2.19) leads to system similar to (2.2.11), solving this system implies

$$
\begin{aligned}
r_1 &= -\tfrac{G+E\omega_2^2}{\omega_1^2-\omega_2^2}, && r_2 = -\tfrac{H+F\omega_2^2}{\omega_1\left(\omega_1^2-\omega_2^2\right)},\\
r_3 &= \tfrac{G+E\omega_1^2}{\omega_1^2-\omega_2^2}, && r_4 = \tfrac{H+F\omega_1^2}{\omega_2\left(\omega_1^2-\omega_2^2\right)}\,.
\end{aligned}
\tag{2.2.20}
$$

Applying the inverse finite Fourier cosine transform to equation (2.2.18) gives the solution of the rotational angle $\psi(x,t)$ in the form

$$
\psi(x,t) = V(0,t) + 2\sum_{n=1}^{\infty}\left(r_1\,\cos(\omega_1\,t) + r_2\,\sin(\omega_1\,t) + r_3\,\cos(\omega_2\,t) + r_4\,\sin(\omega_2\,t)\right)\,\cos(n\pi x),
\tag{2.2.21}
$$

where $r_i(n), i = 1..4.$ are given by equation (2.2.20) and

$$
V(0,t) = -\alpha\,G(0)\cos\left(\frac{t}{\sqrt{\alpha}}\right) - \alpha^{\frac{3}{2}}\,H(0)\,\sin\left(\frac{t}{\sqrt{\alpha}}\right).
\tag{2.2.22}
$$

Similarly, this solution can be written as

$$
\psi(x,t) = V(0,t) + 2\sum_{n=1}^{\infty}\left(\delta_1\,\cos(\mu_1 - \omega_1\,t) + \delta_2\,\cos(\mu_2 - \omega_2\,t)\right)\,\cos(n\pi x),
$$

where $\delta_1 = \sqrt{r_1^2 + r_2^2},\ \delta_2 = \sqrt{r_3^2 + r_4^2},\ \mu_1 = \tan^{-1}\left(\frac{r_2}{r_1}\right)$ and $\mu_2 = \tan^{-1}\left(\frac{r_4}{r_3}\right)$.

Moreover, using equations (2.2.10) and (2.2.19), together with the formulas in section (3.5), results in obtaining $C(n), D(n), G(n)$ and $H(n)$ in term of

$A(n), B(n), E(n)$ and $F(n)$ as follows

$$\begin{aligned} C(n) &= n\pi E(n) - n^2\pi^2 A(n), \\ D(n) &= n\pi F(n) - n^2\pi^2 B(n), \\ G(n) &= \tfrac{1}{\alpha}\left(n\pi A(n) - (1+\beta n^2\pi^2)E(n)\right), \\ H(n) &= \tfrac{1}{\alpha}\left(n\pi B(n) - (1+\beta n^2\pi^2)F(n)\right). \end{aligned} \tag{2.2.23}$$

Table (3.1) below gives the properties of aluminum and polycarbonate used in the computations.

Table 2.1: Properties of Aluminum and Polycarbonate

		Aluminum	Polycarbonate
Density	ρ	2700 kg/m^3	1200 kg/m^3
Shear modulus	G	26.92 GPa	2.3 GPa
Second moment of inertia $I = \frac{bh^3}{12}$	I	$8.33 \times 10^{-10}\ m^4$	$8.33 \times 10^{-10}\ m^4$
Elastic modulus	E	69 GPa	2.6 GPa

2.3 Results and discussion

The Timoshenko system with boundary conditions has been decoupled in the form of fourth order partial differential equation with equivalent boundary conditions in terms of transverse displacement φ. The new initial-boundary value problem (IBVP) has been solved using the finite Fourier sine transformation. Similarly, the system has been written as one equation with its boundary conditions in term of the rotational angle ψ and, later, the finite Fourier cosine transformation has been used to solve it. We have verified that the solutions $\varphi(x,t)$ and $\psi(x,t)$ of the first and second problems are indeed the solutions of the original system.

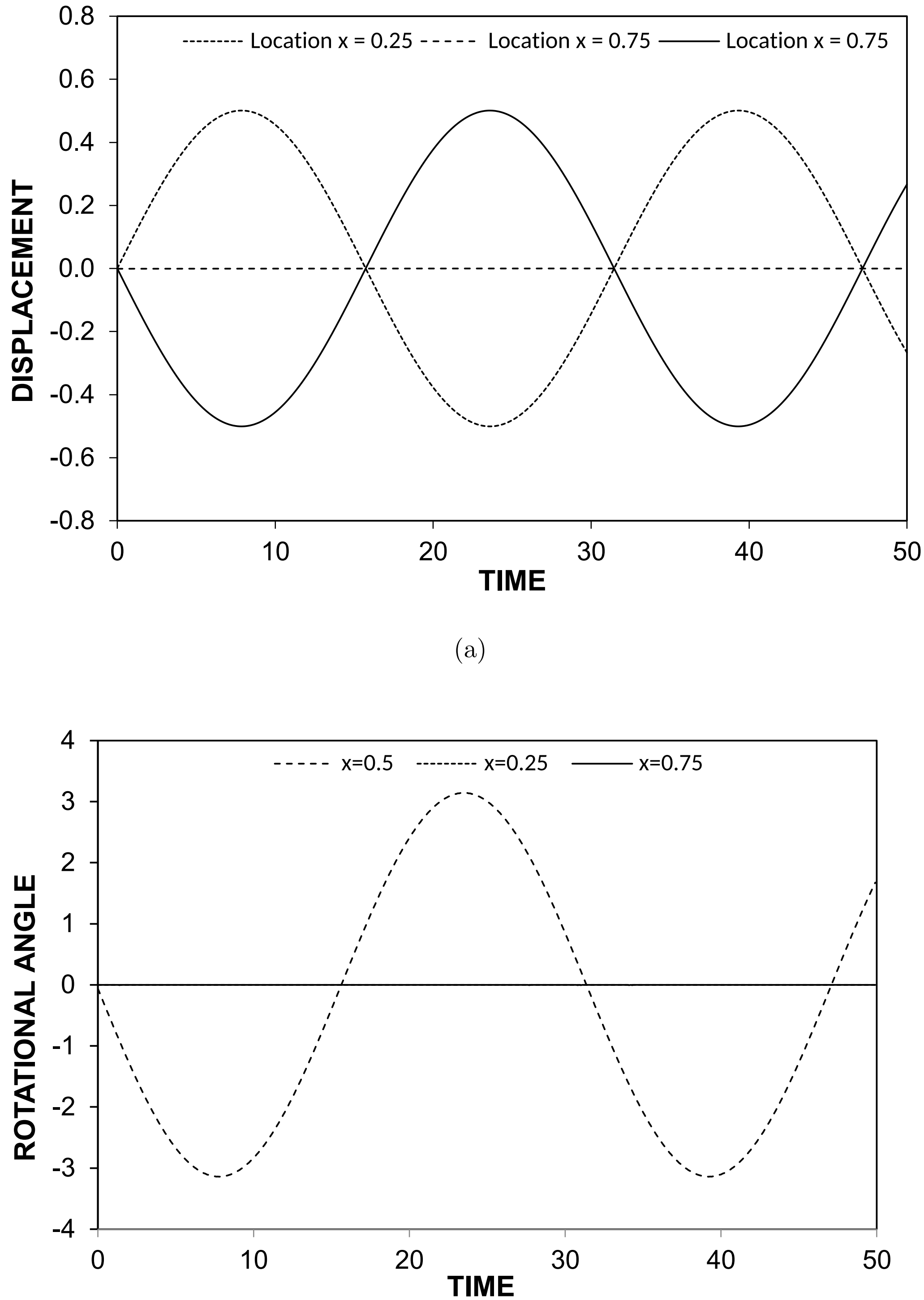
To examine the effect of beam material on the displacement and rotational angle, the temporal and spatial responses of the aluminum and polycarbonate beam with a rectangular cross section ($0.01 \times 0.01\ m^2$) and corresponding Timoshenko shear coefficient $\kappa = \frac{5}{6}$, are presented for the conditions of:

$$\varphi_0(x) = 0, \qquad \varphi_1(x) = \tfrac{1}{10}\sin(2\pi x),$$

and

$$\psi_0(x) = 0, \qquad \psi_1(x) = \tfrac{1}{10}\cos(2\pi x),$$

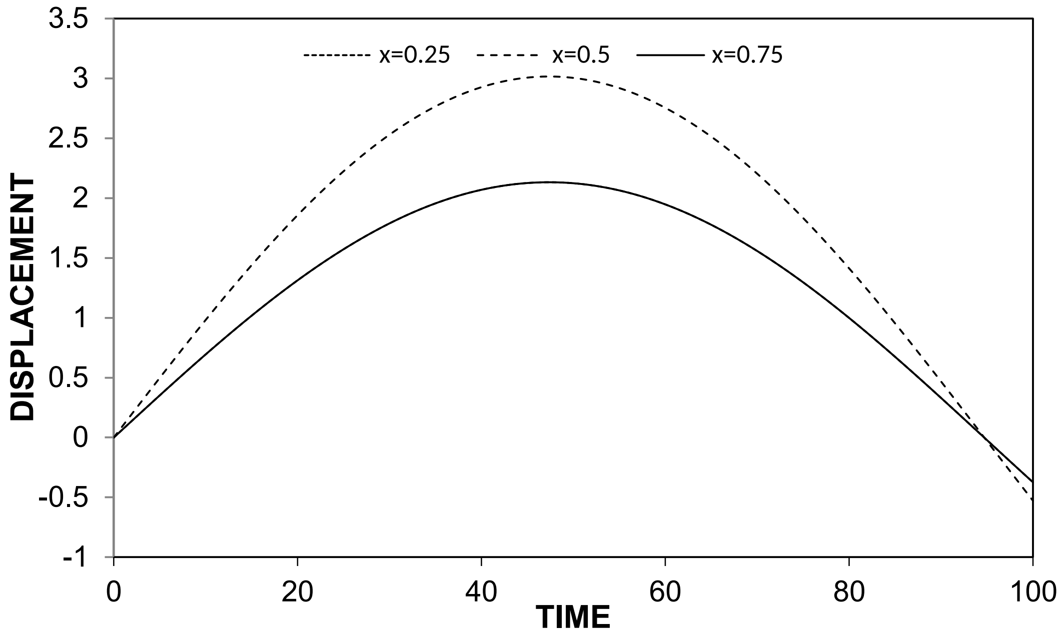
Figures 2.1 and 2.2 below, show the temporal variation of displacement and angular rotation of the aluminum and polycarbonate beam for various locations along the beam length. Here, x represents the dimensionless length ($\frac{\bar{x}}{L}$, where is $\bar{x}$ the dimensional length scale and L is the length of the beam) of the beam, *i.e.* $x = 0.5$ is the midsection of the beam. In the case of aluminum beam, the displacement remains zero at the midpoint of the beam; however, it demonstrates oscillatory behavior at the x-axis locations $x = 0.25$ and $x = 0.75$. The occurrence of the peak values of displacement and rotational angle appears to be identical at dimensionless time of 8. Therefore, the shear effect on the beam becomes significant at locations $x = 0.25$ and $x = 0.75$ along the beam length. Since the deflection remains zero at the midpoint for all times, the shear is not significant at this location. In the case of the beam rotation, it does not remain zero at the midpoint of the beam. Therefore, the beam suffers from the torsional stress at the midpoint yet the shear effect due to beam displacement is zero. Moreover, the occurrence of the peak displacement and peak torsion in a cyclic manner can create fatigue effect on the beam material. In the case of the polycarbonate beam, displacement and torsion characteristics of the beam differ significantly from the aluminum beam Figure 2.1. In this case, the displacement at $x = 0.25$ and $x = 0.75$ differs for the polycarbonate beam; however, the displacement remains zero at the beam midpoint. The occurrence of the maximum rotation and deflection becomes same for the polycarbonate beam. When comparing the occurrence of the maximum displacement and rotation with its counterpart corresponding to the aluminum beam, the occurrence of the maximum displacement happens to be delayed for the polycarbonate beam. Therefore, the beam response, due to external loading, depends on the beam material properties. In this case, the shift in the maximum displacement and torsion is associated with the beam material properties.



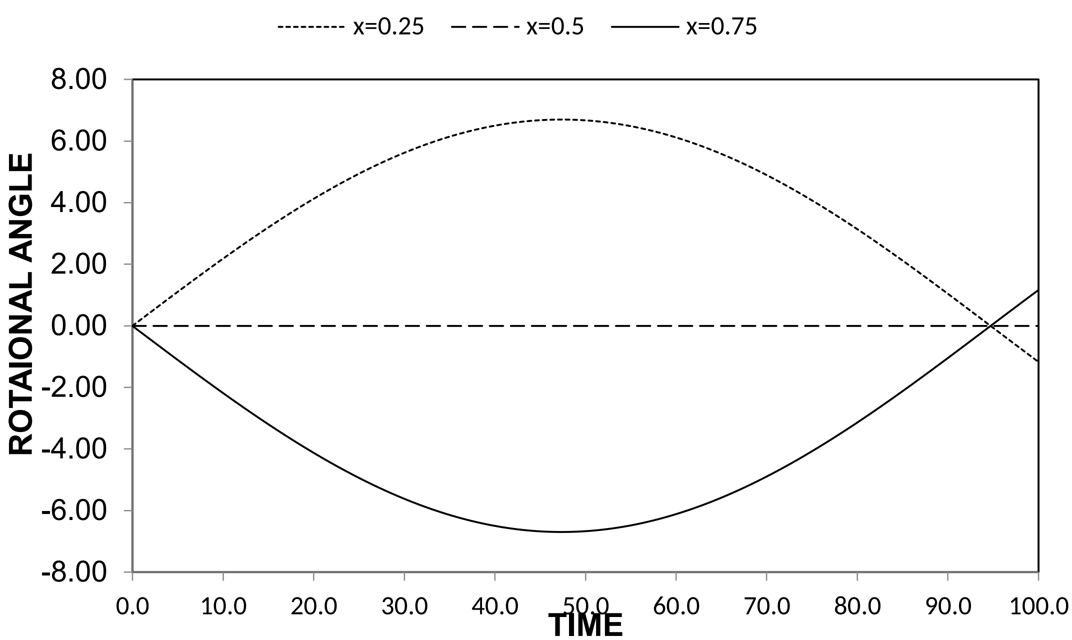
(a)

(b)

Figure 2.1: Temporal variation of displacement and angular rotation of aluminum beam.



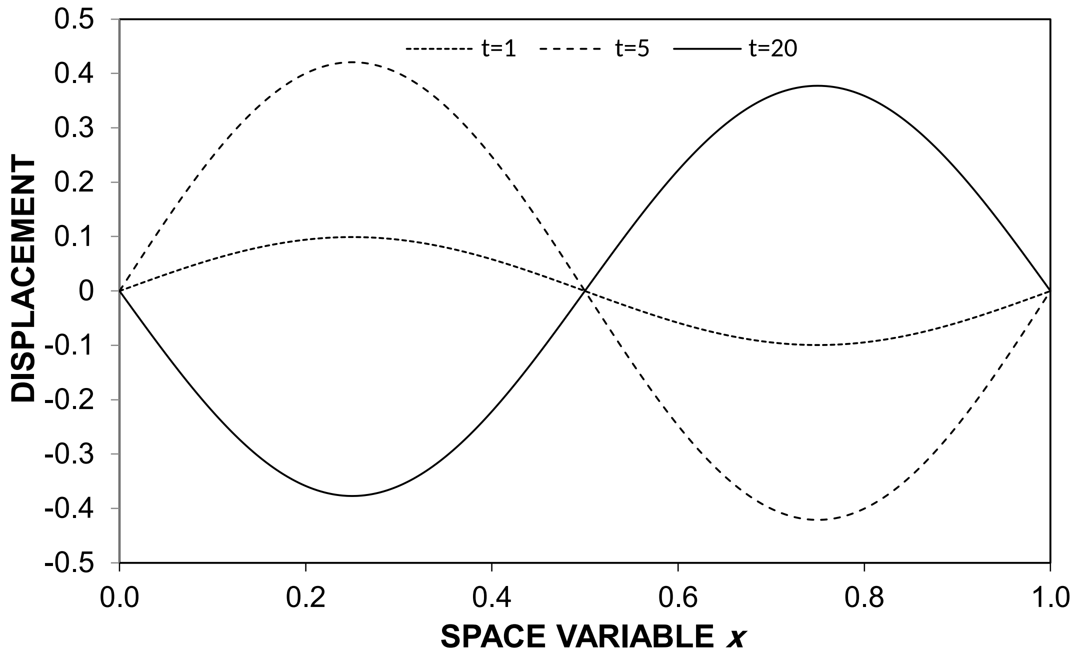
(a)



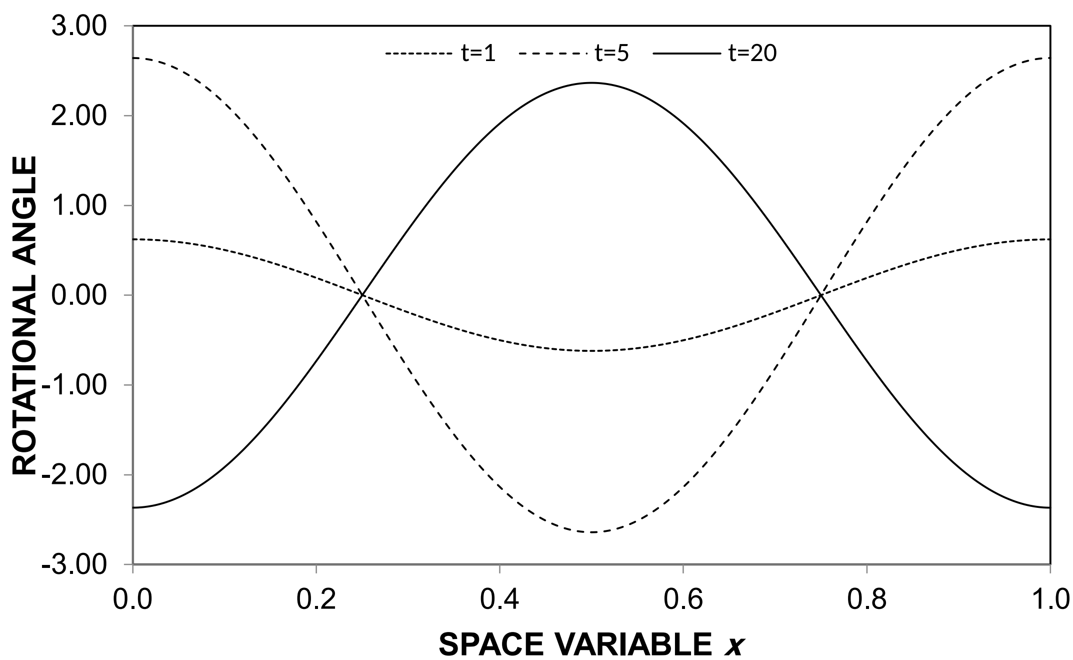
(b)

Figure 2.2: Temporal variation of displacement and angular rotation of polycarbonate beam.

Figures 2.3 and 2.4 below, show displacement and angular rotation along the beam length for the aluminum and the polycarbonate beams at different durations. In the case of aluminum beam, the magnitude of the displacement changes with time at locations $x = 0.25$ and $x = 0.75$, provided that the displacement changes its direction at these locations along the beam. The rotational behavior of the beam reveals that the angle of rotation increases with progressing time and the maximum torsion takes place at the midpoint of the beam length. Consequently, torsional vibration becomes important at the midpoint of the beam; however, shear effect due to the beam displacement becomes significant at locations $x = 0.25$ and $x = 0.75$ along the beam. In the case of the polycarbonate beam figure 2.4, the mode of displacement and torsion changes significantly from the aluminum beam. In this case, the maximum displacement occurs at the midpoint of the beam, unlike the aluminum beam. The angular rotation of the beam remains zero at the midpoint of the beam for all times. The location of the maximum shear, due to displacement, and the maximum torsion on the beam differs for both materials. Therefore, when design the structures with both ends fixed and undergoing bending and torsion, the material properties need to be considered to evaluate the location and occurrence of the maximum displacement and the torsion in the beam.

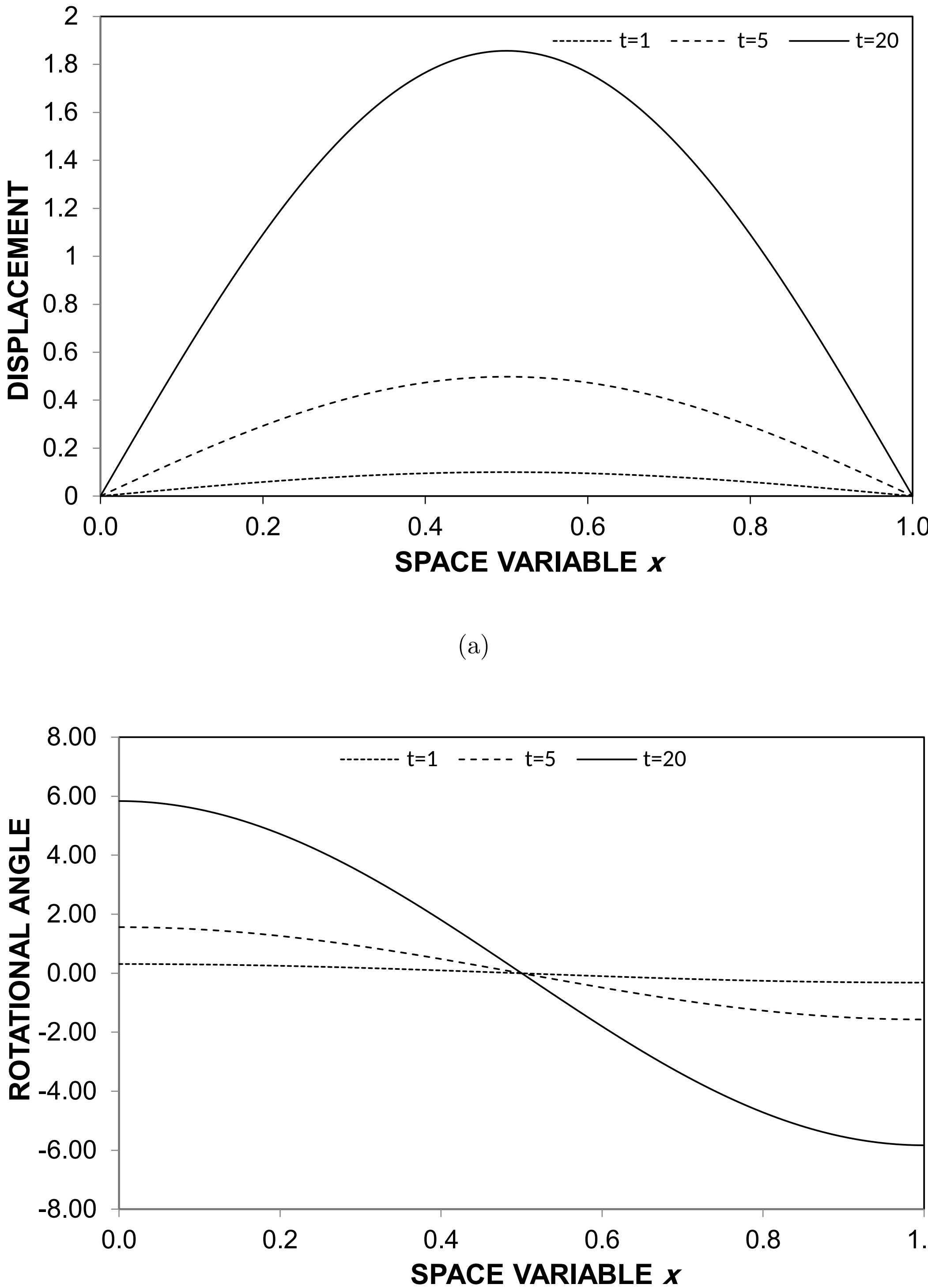


(a)



(b)

Figure 2.3: Variation of displacement and angular rotation along the x-axis for aluminum beam.



(a)

(b)

Figure 2.4: Variation of displacement and angular rotation along the x-axis for polycarbonate beam.

2.4 Conclusion

The analytical solution of the Timoshenko beam system with the appropriate boundary conditions is presented. The system equations have been decoupled prior to the solutions and the boundary conditions for the transverse displacement φ and the angular rotation are introduced in the form of periodic functions. The new initial-boundary value problem (IBVP) has been solved after incorporating the finite Fourier sine transformation. The response of the Timoshenko beam system has been formulated in term of the rotational angle ψ. Later, the finite Fourier cosine transformation has been used to obtain the closed form solution. We have verified that the solutions for displacement $\varphi(x,t)$ and rotational angle $\psi(x,t)$ are for the first and second problems are verified that the problems are the solution of the original system. The location of the occurrence of the maximum displacement and rotational angle changes with the material properties of the beam. In this case, aluminum beam responds to the external loading early for the displacement and rotation. In the region of high displacements, the beam suffers from the shear effect and its location along the beam length changes with the beam material properties. Similar arguments are also true for the torsional response of the beam. Consequently, when designing the structure with both ends fixed (hinged-hinged configuration) and subjected to the external loading, mimicking the Timoshenko beam system, material properties play critical role in terms of the maximum shear and torsion resulted in the beam.

2.5 Finite Fourier sine and cosine transform formulas

Following are finite Fourier sine and cosine transforms of displacement and angular rotation equations.

The finite Fourier sine transform for $\varphi(x,t)$ and its derivatives with respect to x are:

$$\begin{aligned}
F_s\left\{\varphi(x,t),x,n\right\} &= \int_0^L \varphi(x,t)\,\sin\left(\tfrac{n\pi x}{L}\right)\,dx = U(n,t),\\
F_s\left\{\varphi_x(x,t),x,n\right\} &= -\left(\tfrac{n\pi}{L}\right)F_C\left\{\varphi,x,n\right\},\\
F_s\left\{\varphi_{xx}(x,t),x,n\right\} &= -\left(\tfrac{n\pi}{L}\right)^2 F_s\left\{\varphi,x,n\right\} + \left(\tfrac{n\pi}{L}\right)\left(\varphi(0,t)+(-1)^{n+1}\varphi(L,t)\right),\\
F_s\left\{\varphi_{xxxx}(x,t),x,n\right\} &= \left(\tfrac{n\pi}{L}\right)^4 F_s\left\{\varphi,x,n\right\} + \left(\tfrac{n\pi}{L}\right)\left(\varphi_{xx}(0,t)+(-1)^{n+1}\varphi_{xx}(L,t)\right)\\
&\quad -\left(\tfrac{n\pi}{L}\right)^3\left(\varphi(0,t)+(-1)^{n+1}\varphi(L,t)\right),
\end{aligned}$$

and the inverse finite Fourier sine given by

$$\varphi(x,t) = \tfrac{2}{L}\sum_{n=1}^{\infty} U(n,t)\sin\left(\tfrac{n\pi x}{L}\right).$$

The finite Fourier cosine transform for $\psi(x,t)$ and its derivatives with respect to x are

$$
\begin{aligned}
F_C\left\{\psi(x,t),x,n\right\} &= \int\limits_0^L \psi(x,t)\,\cos\left(\tfrac{n\pi x}{L}\right)\,dx = V(n,t),\\
F_C\left\{\psi_x(x,t),x,n\right\} &= \left(\tfrac{n\pi}{L}\right)F_s\left\{\psi,x,n\right\} - \left(\psi(0,t)+(-1)^{n+1}\,\psi(L,t)\right),\\
F_C\left\{\psi_{xx}(x,t),x,n\right\} &= -\left(\tfrac{n\pi}{L}\right)^2 F_C\left\{\psi,x,n\right\} - \left(\psi_x(0,t)+(-1)^{n+1}\,\psi_x(L,t)\right),\\
F_C\left\{\psi_{xxxx}(x,t),x,n\right\} &= \left(\tfrac{n\pi}{L}\right)^4 F_C\left\{\psi,x,n\right\} - \left(\psi_{xxx}(0,t)+(-1)^{n+1}\,\psi_{xxx}(L,t)\right)\\
&\quad + \left(\tfrac{n\pi}{L}\right)^2 \left(\psi_x(0,t)+(-1)^{n+1}\,\psi_x(L,t)\right)
\end{aligned}
$$

and the inverse finite Fourier cosine given by

$$\psi(x,t) = \tfrac{1}{L}V(0\,,t) + \tfrac{2}{L}\sum_{n=1}^{\infty} V(n,t)\,\cos\left(\tfrac{n\pi x}{L}\right).$$

For more details, the reader should refer to [16].

CHAPTER 3

HINGED- HINGED LINEAR TIMOSHENKO BEAM SYSTEM WITH TWO WEAK DAMPING

Analytical treatment of flexural and torsional characteristics of hinged-hinged Timoshenko system is carried out and the solution for the dynamic response of the beam due to external excitation is presented. In order to assess damping effect on flexural and torsional characteristics of hinged-hinged beam, the aspect ratio of the beam cross-section is changed while keeping the beam cross-sectional area constant. This arrangement provides the change of the second moment of an area and the damping factor of the beam. Two beam materials, namely concrete and steel, are incorporated in analysis to resemble the high and low damping materials. It is found that introducing the damping factor of the beam material for flexural motion increases logarithmic decay of flexural and torsional oscillations, despite torsional motion is considered to be elastic. Reducing the aspect ratio of the beam cross-section lowers logarithmic decay of the amplitude of flexural and torsional oscillations, which is more pronounced for the steel beam. In addition, this arrangement modifies the damping frequency of the oscillations.

3.1 Introduction

Many structures, such as bridges and columns, resemble hinged-to-hinged like beams and suffer from mechanical oscillating loads. Some of these structures show similar behavior of Timoshenko system with damping [46]. In this case, hinged-to-hinged beam like structures may undergo torsional and flexural motions such as those observed in natural disasters involving earthquakes, strong winds, water hammer, and etc. Simplifying of such structures by a Timoshenko system may not exactly match the actual features of the beam; however, the dynamic analysis of the beam provides useful information about the dynamic response in terms of torsional and flexural characteristics of the structure under the damping conditions. Consequently, investigation of the dynamic response of hinged-hinged Timoshenko beam with damping factor while resembling a concrete or a steel beam becomes essential.

Considerable research studies were carried out to examine mechanical response of a Timoshenko beam incorporating the damping factor of the material. The stability of Mindlin-Timoshenko plate with nonlinear boundary damping and boundary sources was studied by Pei [60]. He proposed the well-posedness and stability model for Mindlin-Timoshenko plates under the interplay of damping and source terms acting either in the interior or on the boundary of the plate. Structural damping in Timoshenko beams on elastic foundations under moving loads was investigated by Ding et al. [19]. They determined the wave velocity pass bands and stop bands through calculating the localization factor. In addition, they analyzed the interactions between the static axial load and moving load. Elastic waves in a Timoshenko beam with boundary damping were examined by Roux et al. [66]. They showed that the boundary moment had a significantly effect on lowering vibrations in the beam; in which case, the second spectrum of the Timoshenko beam played a prominent part in explaining this phenomenon. The forced vibration analysis of bending-torsion coupled Timoshenko beam was carried out by Han et al. [27]. They demonstrated that Green's functions for the Timoshenko beam could be reduced to those for Euler-Bernoulli beam by setting the values of shear rigidity and rotational inertia. In addition, they noted that coupling effects between bending and torsional vibrations of the beam could be analyzed through the analytical Green's functions and the direct expressions of the steady-state responses with various loadings could be obtained by using the superposition principle. The damping effects in Timoshenko beam were studied by Capsoni et al. [9]. They analyzed the dynamic response of a Timoshenko beam with distributed internal viscous damping with the aim to ascertain their relative effects on the whole range of beam slenderness. The parametric study on bending vibration of axially-loaded twisted Timoshenko beam with locally distributed Kelvin-Voigt damping was carried out by Chen [11]. He used a finite element method to reduce the equation of motion into linear second-order ordinary differential equation with constant coefficients. Moreover, a quadratic eigenvalue problem of a damped system was formulated to examine the effects of the twist angle, damping amount, size and location of damped segment, axial load and restraint types on the eigen-frequency of the damped twisted beams. Analysis of the thermal effect on vibrations of a damped Timoshenko beam was carried out by Gu et al. [26]. The findings revealed that the thermal eigenvalues should be considered together with the cross-sectional temperature gradients in predicting thermal vibrations. The results also highlighted the role of axial temperature gradients on the dynamic response induced by equivalent shearing forces. Indirect stabilization of a Mindlin-Timoshenko plate was examined by Tebou [77]. He showed that when the plate was clamped (Dirichlet boundary conditions), the speed of propagation of the wave generated by the longitudinal component of the rotation angle and that of the wave generated by the vertical deflection was identical. Analytical and numerical study for thick beams with thermoelastic damping was carried out by Parayil [57]. He indicated that the analytical solution based on a Timoshenko beam model gave a better result when compared to the analytical model based on the Euler-Bernoulli model. Numerical solutions were in good agreement with both analytical results and three-dimensional finite element results when the aspect ratio (image) was high. In addition, the heat transfer in the axial direction could not be ignored while computing quality factor attributable to thermoelastic damping in thick beams. The generalized function approach for the frequency response analysis of beams and plane frames with the inclusion of viscoelastic damping was introduced by Failla [22]. He indicated that from the nodal displacement solution, the exact frequency response in all frame members could be obtained in a closed analytical form. Timoshenko systems with indefinite damping were studied by Rivera and Racke [62]. They considered the Timoshenko system in a bounded domain image and the system had an indefinite damping mechanism. They demonstrated that the system was exponentially stable under the positive constant damping condition. Exponential decay of a viscoelastically damped Timoshenko beam was investigated by Tatar [75]. He demonstrated that for a non-decreasing function "Gamma" whose "logarithmic derivative" was decreasing to zero, a decay of order Gamma to some power occurred. In the case that it decreased to a different value than zero, then the decay was exponential. Dynamic responses of a hinged-hinged Timoshenko beam with and without a damage subject to blast loading was examined by Metsebo et al. [45]. They derived the general equation governing the dynamic states of both healthy and damaged hinged-hinged Timoshenko beam and they proposed a damage function evolving the location, the sizes, and the geometry of the damaged area. The dynamic analysis of damping in layered and welded beams was carried out by Singh and Nanda [69]. They assumed the Bernoulli-Euler hypothesis and a linear constitutive expression between the horizontal slip and the interfacial shear force were developed to quantify the amount of energy dissipation during dynamic loading.

Although dynamic response of hinged-hinged Timoshenko system was studied previously [45], the main focus was evaluating the damage function. However, flexural and torsional characteristics of the beam with damping factor were left for the future study. In addition, the cross-sectional dimension of the beam is critical in terms of flexural and rotational motion of the beam when external excitation is introduced. Consequently, in the present study, analytical solution for the Timoshenko system resembling hinge-hinge beam incorporating with and without the damping factor. The torsional and flexural characteristics, in terms of displacement and frequency, are analyzed in line with the cyclic boundary conditions. In order to assess the beam material on the dynamic response of the hinge-hinge beam, steel and concrete are considered as the beam material.

3.2 Mathematical analysis

In this paper, we consider the initial-boundary value problem (IBVP) representing a hinged-hinged Timoshenko beam with length L and subjected to two weak damping. The equation of motion is modeled by the following system of partial differential equations (PDEs):

$$
\begin{aligned}
&\rho A\bar{\varphi}_{\bar{t}\bar{t}} + AG\kappa\left(\bar{\psi} - \bar{\varphi}_{\bar{x}}\right)_{\bar{x}} + \tfrac{\bar{d_1}}{L}\,\bar{\varphi}_{\bar{t}} = 0, && (0,L)\times\mathbb{R}^+,\\
&\rho I\bar{\psi}_{\bar{t}\bar{t}} - EI\bar{\psi}_{\bar{x}\bar{x}} + AG\kappa\left(\bar{\psi} - \bar{\varphi}_{\bar{x}}\right) + \tfrac{\bar{d_2}}{L}\,\bar{\psi}_{\bar{t}} = 0, && (0,L)\times\mathbb{R}^+,\\
&\bar{\varphi}(0,\bar{t}) = \bar{\varphi}(L,\bar{t}) = \bar{\psi}_{\bar{x}}(0,\bar{t}) = \bar{\psi}_{\bar{x}}(L,\bar{t}) = 0, && \bar{t}\geq 0,\\
&\bar{\varphi}(\bar{x},0) = \bar{\varphi}_0(\bar{x}), \quad \bar{\varphi}_{\bar{t}}(\bar{x},0) = \bar{\varphi}_1(\bar{x}), && \bar{x}\in(0,L),\\
&\bar{\psi}(\bar{x},0) = \bar{\psi}_0(\bar{x}), \quad \bar{\psi}_{\bar{t}}(\bar{x},0) = \bar{\psi}_1(\bar{x}), && \bar{x}\in(0,L),
\end{aligned}
\tag{3.2.1}
$$

where $\bar{x}$ denotes the space variable along the beam of length L, $\bar{t}$ is the time variable, $\bar{\varphi}(\bar{x},\bar{t})x$ is the transverse displacement of the beam and $\bar{\psi}(\bar{x},\bar{t})x$ is the rotational angle of the filament of the beam. The Timoshenko model takes into account the effect of shear as well as the effect of rotation to the Euler-Bernoulli beam model. Also $\bar{\psi} - \bar{\varphi}_{\bar{x}}$ represents the shear angle, and the terms $\bar{d_1}$ is the bending damping coefficient, and $\bar{d_2}$ is the rotational damping coefficient. The physical parameters appearing in (Eq. 3.2.1) are ρ- the density of the beam material, A- the cross section area, G- the shear modulus, κ- the Timoshenko shear coefficient which is equal $\frac{5}{6}$ for rectangular cross section, I- the second moment of area, E- the elastic modulus.

The dimensionless solution can describe many dimensional solutions. Non-dimensional problems are easier to recognize when mathematical techniques applied, as well such formulation gives insight into what might be small parameters that could be ignored or treated approximately. The dimensionless form not only enhances the usefulness of results but also makes dealing with the problem easier, which is our purpose for writing our model in non-dimensional form.

In order to rewrite (Eq. 3.2.1) in the dimensionless form, define the dimensionless variables

$$t = \frac{\bar{t}}{T}, \quad x = \frac{\bar{x}}{L}, \quad \varphi(x,t) = \frac{\bar{\varphi}(\bar{x},\bar{t})}{L}, \quad \psi(x,t) = \bar{\psi}(\bar{x},\bar{t}),$$

and the dimensionless constants

$$\alpha = \frac{I}{AL^2}, \quad \beta = \frac{EI}{AG\kappa L^2}, \quad d_1 = \frac{L\,\overline{d_1}}{AG\kappa T}, \quad d_2 = \frac{\overline{d_2}}{AG\kappa LT},$$

where the time scaling factor $T = L\sqrt{\frac{\rho}{G\kappa}}$.

Writing system (Eq. 3.2.1) using the dimensionless variables gives the dimensionless form as

$$\begin{aligned}
&\varphi_{tt} - \varphi_{xx} + \psi_x + d_1\varphi_t = 0, && (0,1)\times\mathbb{R}^+, \\
&\alpha\psi_{tt} - \beta\psi_{xx} - \varphi_x + \psi + d_2\psi_t = 0, && (0,1)\times\mathbb{R}^+, \\
&\varphi(0,t) = \varphi(1,t) = \psi_x(0,t) = \psi_x(1,t) = 0, && t \geq 0, \\
&\varphi(x,0) = \varphi_0(x), \qquad \varphi_t(x,0) = \varphi_1(x), && x \in (0,1), \\
&\psi(x,0) = \psi_0(x), \qquad \psi_t(x,0) = \psi_1(x), && x \in (0,1),
\end{aligned} \tag{3.2.2}$$

where $\varphi_0(x), \varphi_1(x), \psi_0(x), \psi_1(x)$ are arbitrary functions satisfying the conditions

$$\varphi_0(0) = \varphi_0(1) = \varphi_1(0) = \varphi_1(1) = \psi_0'(0) = \psi_0'(1) = \psi_1'(0) = \psi_1'(1) = 0.$$

To solve (Eq. 3.2.2), let

$$\begin{aligned}\varphi(x,t) &= X_1(x)\; T_1(t),\\ \psi(x,t) &= X_2(x)\; T_2(t),\end{aligned} \tag{3.2.3}$$

Substituting (Eq. 3.2.3) in (Eq. 3.2.2) gives

$$\begin{aligned} X_1\, \ddot{T}_1 - X_1''\; T_1 + X_2'\; T_2 + d_1\, X_1\, \dot{T}_1 = 0,\\ \alpha\, X_2\, \ddot{T}_2 - \beta\, X_2''\, T_2 - X_1'\, T_1 + X_2\, T_2 + d_2\, X_2\, \dot{T}_2 = 0.\end{aligned} \tag{3.2.4}$$

Dividing the first equation by $\varphi(t,x)$ and the second equation by $\psi(t,x)$ implies

$$\begin{aligned} \tfrac{\ddot{T}_1}{T_1} - \tfrac{X_1''}{X_1} + \tfrac{X_2'\, T_2}{X_1\, T_1} + d_1\, \tfrac{\dot{T}_1}{T_1} = 0,\\ \alpha\, \tfrac{\ddot{T}_2}{T_2} - \beta\, \tfrac{X_2''}{X_2} - \tfrac{X_1'\, T_1}{X_2\, T_2} + 1 + d_2\, \tfrac{\dot{T}_2}{T_2} = 0.\end{aligned} \tag{3.2.5}$$

In previous study Han, et al. [28], they considered a solution of (Eq. 3.2.2) without damping term using the separation of variables method under the assumption that $\varphi(t,x)$ and $\psi(t,x)$ share the same time solution, (i.e. $T_1(t) = T_2(t)$).

In the current work, we consider the solution of the IBVP (Eq. 3.2.2) using separation of variables method under the assumption that, $\varphi(t,x)$ and $\psi(t,x)$ share the ratio between the second derivative of the space part and itself, as follows

$$\tfrac{X_1''}{X_1} = \tfrac{X_2''}{X_1} = -\Omega^2. \tag{3.2.6}$$

The existence of the solution of the well-posed IBVP (Eq. 3.2.2) [71, 64] under the assumptions (Eq. 3.2.6) proves its validity.

Solving (Eq. 3.2.6) gives

$$\begin{aligned} X_1(x) &= C_1\, \cos(\Omega x) + \, C_2\, \sin(\Omega x)\,, \\ X_2(x) &= C_3\, \cos(\Omega x) + \, C_4\, \sin(\Omega x)\,. \end{aligned} \tag{3.2.7}$$

The boundary conditions in (Eq. 3.2.2) implies

$$\begin{aligned} X_1(0) = 0, &\qquad X_1(1) = 0, \\ X_2'(0) = 0, &\qquad X_2'(1) = 0. \end{aligned} \tag{3.2.8}$$

Applying conditions (Eq. 3.2.8) to equation (Eq. 3.2.7) yields the nontrivial solution

$$\begin{aligned} X_1(x, n) &= \sin(n\pi x), \\ X_2(x, n) &= \cos(n\pi x), \end{aligned} \tag{3.2.9}$$

corresponding to $\Omega = n\Pi$, where n is non-negative integer number.

The assumptions (Eq. 3.2.6) reduces the nonlinear (Eq. 3.2.5) to the following linear system

$$\begin{aligned} \ddot{T}_1 + d_1\, \dot{T}_1 + \, n^2\, \pi^2\, T_1 - n\, \pi\, T_2 = 0, \\ \alpha\, \ddot{T}_2 + d_2\, \dot{T}_2 + \, (1 + \beta\, n^2\, \pi^2)\, T_2 \, - n\, \pi\, T_1 \, = 0. \end{aligned} \tag{3.2.10}$$

Thus, by superposition principle, the transverse displacement $\varphi(x, t)$ and the rotational angle $\psi(x,t)$ solutions are given by

$$\begin{aligned}\varphi(x,t) &= \sum_{n=1}^{\infty} T_1(t,n)\ \sin(n\pi x)\,,\\ \psi(x,t) &= \sum_{n=0}^{\infty} T_2(t,n)\ \cos(n\pi x)\,,\end{aligned} \tag{3.2.11}$$

where $T_1(t,n)$ and $T_2(t,n)$ are the solution of the (Eq. 3.2.10).

Solving the second equation in (Eq. 3.2.10) for $n=0$ provides the solution

$$T_2\,(t,0) = R_1\,\mathrm{e}^{-\left(\frac{d_2-\sqrt{d_2^2-4\,\alpha}}{2\alpha}\right)t} + R_2\,\mathrm{e}^{-\left(\frac{d_2+\sqrt{d_2^2-4\,\alpha}}{2\alpha}\right)t}, \tag{3.2.12}$$

where R_1 and R_2 are arbitrary constants.

For $n \neq 0$, the first equation of (Eq. 3.2.10) can be given in term of T_2 as

$$T_2 = \tfrac{1}{n\pi}\left(\ddot{T}_1 + d_1\,\dot{T}_1 + n^2\pi^2\,T_1\right). \tag{3.2.13}$$

Substituting (Eq. 3.2.13) in the second equation of (Eq. 3.2.10) gives the following fourth order ODE in term of $T_1(t)$

$$\alpha\ \ddddot{T}_1 + \Gamma_1\ \dddot{T}_1 + \Gamma_2\ \ddot{T}_1 + \Gamma_3\ \dot{T}_1 + \beta\, n^4\pi^4 T_1 = 0, \tag{3.2.14}$$

where $\Gamma_1 = \alpha\, d_1 + d_2$, $\Gamma_2 = 1 + d_1\, d_2 + (\alpha+\beta)\, n^2\pi^2$ and $\Gamma_3 = d_1 + (\beta\, d_1 + d_2)\, n^2\pi^2$.

The corresponding characteristic equation is

$$\alpha\, M^4 + \Gamma_1\; M^3 \,+\, \Gamma_2\, M^2 \,+\, \Gamma_3\, M +\, \beta\, n^4\, \pi^4 \;= 0. \tag{3.2.15}$$

In order to solve and know the nature of the quartic equation (Eq. 3.2.15), we used the formula in appendix which is deduced from Ferrari-Lagrange's method [82, 49]. All possible cases for the nature of the roots can be determined according to the previous studies [49, 61]. The discriminant of the quartic equation in the current study is never zero. Therefore, the four roots of (Eq. 3.2.15) are distinct, which means $T_1(t,n)$ has the form

$$T_1(t,n) = C_1\, \mathrm{e}^{\mu_1\, t} + C_2\, \mathrm{e}^{\mu_2\, t} + C_3\, \mathrm{e}^{\mu_3\, t} + C_4\, \mathrm{e}^{\mu_4\, t}, \tag{3.2.16}$$

where $\mu_i, i = 1, ..., 4$ are distinct roots (Eq. 3.2.15).

Substituting back (Eq. 3.2.16) in (Eq. 3.2.13) gives the solution of $T_2(t,n)$ for $n \geq 1$ as

$$\begin{aligned} T_2(t,n) =\; & C_1\left(n\pi + \tfrac{\mu_1}{n\pi}\,(d_1 + \mu_1)\right)\, \mathrm{e}^{\mu_1\, t} + C_2\left(n\pi + \tfrac{\mu_2}{n\pi}\,(d_1 + \mu_2)\right)\, \mathrm{e}^{\mu_2\, t} \\ & + C_3\left(n\pi + \tfrac{\mu_3}{n\pi}\,(d_1 + \mu_3)\right)\, \mathrm{e}^{\mu_3\, t} + C_4\left(n\pi + \tfrac{\mu_4}{n\pi}\,(d_1 + \mu_4)\right)\, \mathrm{e}^{\mu_4\, t}. \end{aligned} \tag{3.2.17}$$

Now, the initial conditions in (Eq. 3.2.2) are used with the orthogonality to find the following relations among the constants for positive integer n as

$$
\begin{aligned}
T_1(0,n) &= 2\int_0^1 \sin(n\pi x)\,\phi_0(x)\,dx \equiv A(n)\,,\\
T_1'(0,n) &= 2\int_0^1 \sin(n\pi x)\,\phi_1(x)\,dx \equiv B(n)\,,\\
T_2(0,n) &= 2\int_0^1 \cos(n\pi x)\,\psi_0(x)dx \equiv C(n)\,,\\
T_2'(0,n) &= 2\int_0^1 \cos(n\pi x)\,\psi_1(x)dx \equiv D(n)\,,
\end{aligned}
\tag{3.2.18}
$$

and for $n = 0$ as

$$
\begin{aligned}
T_2(0,0) &= \int_0^1 \psi_0(x)\,dx \equiv F\,,\\
T_2'(0,0) &= \int_0^1 \psi_1(x)\,dx \equiv H\,.
\end{aligned}
\tag{3.2.19}
$$

For $n = 0$, R_1 and R_2 are given using (Eq. 3.2.19) as

$$
R_1 = \frac{\left(\sqrt{d_2^2-4\,\alpha}+d_2\right)F+2\,\alpha H}{2\sqrt{d_2^2-4\,\alpha}}\;,\quad R_2 = \frac{\left(\sqrt{d_2^2-4\,\alpha}-d_2\right)F-2\,\alpha H}{2\sqrt{d_2^2-4\,\alpha}}. \tag{3.2.20}
$$

For $n \geq 1$, equations (Eq. 3.2.18) leads to the following values of the constants $C_i(n), i = 1, ..., 4$

$$C_1(n) = \frac{(A(d_1+\mu_2+\mu_3+\mu_4)-B)n^2\pi^2-(C(d_1+\mu_2+\mu_3+\mu_4)-D)n\pi+((\mu_3+\mu_4+d_1)\mu_2+(d_1+\mu_4)(d_1+\mu_3))B-A\mu_2\mu_3\mu_4}{(\mu_1-\mu_2)(\mu_1-\mu_3)(\mu_1-\mu_4)},$$

$$C_2(n) = \frac{-(A(d_1+\mu_1+\mu_3+\mu_4)-B)n^2\pi^2+(C(d_1+\mu_1+\mu_3+\mu_4)-D)n\pi-((\mu_3+\mu_4+d_1)\mu_1+(d_1+\mu_4)(d_1+\mu_3))B+A\mu_1\mu_3\mu_4}{(\mu_1-\mu_2)(\mu_2-\mu_3)(\mu_2-\mu_4)},$$

$$C_3(n) = \frac{(A(d_1+\mu_1+\mu_2+\mu_4)-B)n^2\pi^2-(C(d_1+\mu_1+\mu_2+\mu_4)-D)n\pi+((\mu_2+\mu_4+d_1)\mu_1+(d_1+\mu_2)(d_1+\mu_4))B-A\mu_1\mu_2\mu_4}{(\mu_1-\mu_3)(\mu_2-\mu_3)(\mu_3-\mu_4)},$$

$$C_4(n) = \frac{-(A(d_1+\mu_1+\mu_2+\mu_3)-B)n^2\pi^2+n(C(d_1+\mu_1+\mu_2+\mu_3)-D)\pi-((\mu_3+d_1+\mu_2)\mu_1+(d_1+\mu_2)(d_1+\mu_3))B+A\mu_1\mu_2\mu_3}{(\mu_1-\mu_4)(\mu_2-\mu_4)(\mu_3-\mu_4)}. \tag{3.2.21}$$

Finally, the solution of system (Eq. 3.2.2) can be given by truncating the real part of solution (Eq. 3.2.11) with equations (Eq. 3.2.12), (Eq. 3.2.16) and (Eq. 3.2.17).

As an application, the solution (Eq. 3.2.11) of a hinged-hinged Timoshenko system of concrete and steel beam materials are considered incorporating with flexural damping and torsional damping.

The flexural damping coefficient $\bar{d_1}$ is evaluated using the formula $\bar{d_1} = \zeta_1 C_c$, where ζ_1 is the flexural damping ratio, and C_c is the critical flexural damping which can be calculating by using the formula $C_c = 2\sqrt{k_1 M}$, where M is the mass and k_1 is the flexural stiffness which is calculated by $k_1 = \frac{48EI}{L^3}$ for elastic modulus E, second moment of area of a rectangular cross section $I = \frac{bh^3}{12}$, and h, b, L are Height, Width and Length of the beam. The unit of the flexural damping coefficient is $\frac{N.s}{m}$.

The natural (undamped) frequency of flexural mechanical calculated using the formula $\omega_n^1 = \sqrt{\frac{k_1}{M}}$, and the damping frequency for flexural vibration calculated by $\omega^1 = \omega_n^1\sqrt{1-\zeta_1^2}$, both damped and undamped frequency unit is $\frac{rad}{s}$.

The torsional damping coefficient $\bar{d}_2$ is evaluated using the formula $\bar{d}_2 = \zeta_2 C_t$, where ζ_2 is the torsional damping ratio, and C_t is the critical torsional damping which can be calculating by using the formula $C_t = 2\sqrt{k_2 I_*}$, where I_* is the moment of inertia $kg.m^2$ and calculated by $I_* = \frac{M}{2}(h^2+b^2)$, and k_2 is the rotational stiffness which is calculated by $k_2 = \frac{GJ}{L}$ for shear modulus G, and torsional constant J of the cross section which calculating using the formula $J = \beta_*\, hb^3$, where β_* is found according to the corresponding ratio of height to width as it is illustrated in the following table [83].

b/h	1.0	1.5	2.0	2.5	3.0	4.0	5.0	6.0	10.0	∞
β_*	0.141	0.196	0.229	0.249	0.263	0.281	0.291	0.299	0.312	0.333

The torsional damping coefficient unit is $\frac{N.m.s}{\sqrt{rad}}$. The natural (undamped) frequency of torsional mechanical calculated using the formula $\omega_n^2 = \sqrt{\frac{k_2}{I_*}}$, and the damping frequency for torsional vibration calculated by $\omega^2 = \omega_n^2\sqrt{1-\zeta_2^2}$, both damped and undamped frequency unit is $\frac{rad}{s}$.

The physical properties using in the simulations are given in Table 1. The density ρ, shear modulus G and elastic modulus E can easily found in the literature. Flexural and torsional damping ratios for concrete and steel can be found in [8, 87]. In Table 2, the natural and damping frequencies of concrete and steel beams due to both flexural and torsional damping are presented for different values of d_1

and d_2 corresponding to cross-sections with dimensions $b = 0.01,\ h = 0.1$ and $b = 0.02,\ h = 0.05$ for concrete and steel materials.

Table 3.1: Properties used in the simulations

Property		Steel	Concrete
Density kg/m^3	ρ	7840	2400
Shear modulus pa	G	81×10^9	13×10^9
Elastic modulus pa	E	210×10^9	30×10^9
Flexural damping ration	ζ_1	0.0015	0.035
Torsional damping ration	ζ_2	0.001	0.0147

Table 3.2: Natural and damping frequencies of concrete and steel beams due to flexural and torsional damping. The cross-section of the beam modifies the damping factor of the beams.

Material - Concrete	$d_1 = 0.0233$ and $d_2 = 5.24 \times 10^{-6}$ $b = 0.01$ m and $h = 0.1$ m
Flexural Natural Frequency (Hz)	707.12
Torsional Natural Frequency (Hz)	448.1
Flexural Damped Frequency (Hz)	706.7
Torsional Damped Frequency (Hz)	448.05
Material - Concrete	$d_1 = 0.012$ and $d_2 = 5.01 \times 10^{-6}$ $b = 0.02$ m and $h = 0.05$ m
Flexural Natural Frequency (Hz)	353.55
Torsional Natural Frequency (Hz)	1494.13
Flexural Damped Frequency (Hz)	353.34
Torsional Damped Frequency (Hz)	1493.96
Material - Steel	$d_1 = 0.0011$ and $d_2 = 3.55 \times 10^{-7}$ $b = 0.01$ m and $h = 0.1$ m
Flexural Natural Frequency (Hz)	1035.098
Torsional Natural Frequency (Hz)	618.859
Flexural Damped Frequency (Hz)	1035.097
Torsional Damped Frequency (Hz)	618.858
Material - Steel	$d_1 = 0.00053$ and $d_2 = 3.39 \times 10^{-7}$ $b = 0.02$ m and $h = 0.05$ m
Flexural Natural Frequency (Hz)	517.549
Torsional Natural Frequency (Hz)	2063.506
Flexural Damped Frequency (Hz)	517.548
Torsional Damped Frequency (Hz)	2063.505

3.3 Results and discussion

The dynamic response of a hinged-hinged Timoshenko system is considered incorporating the damping of the beam material. The flexural and torsional characteristics of the hinged-hinged Timoshenko beam are analyzed for given sets of dynamic boundary conditions and initial conditions of the form

$$\phi_0(x) = 0, \;\; \phi_1(x) = \tfrac{1}{10} \sin(\pi x),$$

and

$$\psi_0(x) = 0, \;\; \psi_1(x) = \tfrac{1}{10} \cos(\pi x).$$

Influence of beam material on the dynamic characteristics is examined introducing concrete and steel beam materials. In addition, effect of the dimensions of the beam cross-section, while keeping the cross-sectional area constant, on the torsional and flexural displacements and oscillation frequencies of the beam is investigated. The damping and natural frequencies of the flexural and torsional oscillations due to concrete and steel beams are given in Table 2.

Figure (3.1) shows flexural and rotational displacements of the concrete hinged-hinged beam with cross section of width $b = 1$ cm and height $h = 10$ cm and length of 1 m. The results are presented for three cases including: i) no damping in flexural and torsional motions, ii) damping is applied only for the flexural motion while no damping is considered for the torsional motion, and iii) damping is considered for flexural and torsional motions of the beam. In the case of the elastic behavior of the beam, in which the damping is zero, both flexural and torsional motion of the beam follow the regular oscillation pattern. The amplitude and frequency of oscillation remain same and the natural frequency is the only dominant frequency of the oscillation. In the case of the presence of a flexural damping without torsional damping, the amplitude of the flexural and torsional oscillations reduces, despite the fact that torsional damping is avoided in the governing equation. The coupling of flexural and torsional motion of the beam is responsible for the decay of amplitude for the torsional motion. However, the logarithmic decay of amplitude for the flexural oscillation is larger than that corresponding to torsional oscillation. Consequently, the damping is more effective in the flexural motion than that of the torsional motion. As the damping is introduced for flexural and torsional motions, logarithmic decay of amplitude for both oscillations becomes significantly large. In addition, oscillation frequency is dominated by the damping frequency; in which case, the frequency of both oscillations is reduced, which is more pronounced for the torsional oscillation. Consequently, hinged-hinged concrete beam demonstrates different oscillatory behavior when material damping is introduced. In this case, both flexural and torsional oscillation of the beam dies out in a short period of time. However, care must be taken to account for the stress levels, which may cause failure of the beam when the frequency of flexural and torsional excitations is increased. In this case, high cyclic fatigue failure may be the most probable situation in the beam.

$d_1 = d_2 = 0$	$d_1 \neq 0,\ d_2 = 0$	$d_1 \neq 0$, and $d_2 \neq 0$

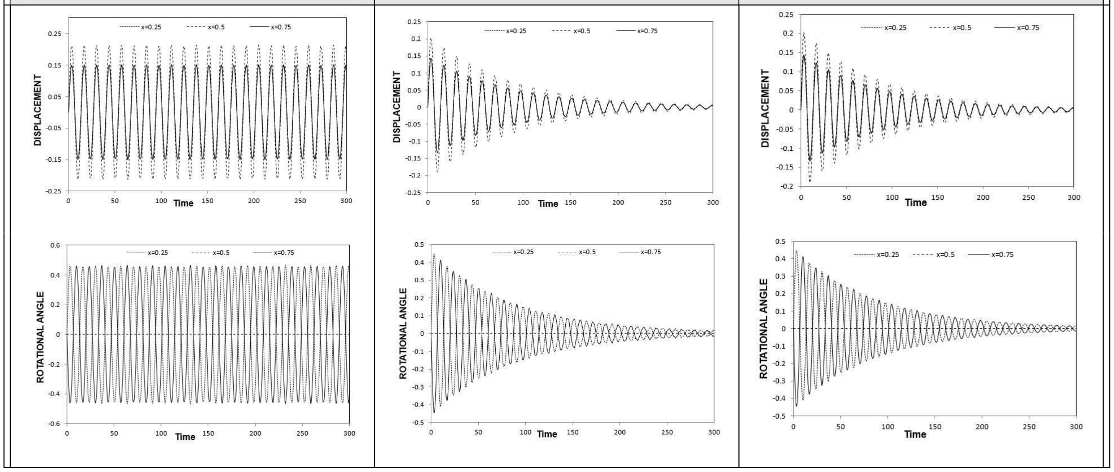


Figure 3.1: Flexural and rotational oscillations of hinged-hinged concrete beam for different damping conditions. The cross-sectional size of the rectangular beam is width (b) = 0.01 m and height (h) = 0.1 m. The length of the beam is 1 m. d1 (0.0233) is the flexural damping factor and d2 (5.24×10^{-6}) is the torsional damping factor of concrete. x- axis locations represent the locations along the beam length.

Figure (3.2) shows hinged-hinged concrete beam with cross-sectional dimension of $b = 2$ cm and $h = 5$ cm, which results in same beam cross-sectional area of the previous case as shown in figure (3.1). Flexural and torsional oscillations of the beam with new cross-sectional size demonstrate similar behavior to those shown in figure (3.1). However, logarithmic decay of amplitude due to flexural and torsional oscillation reduces considerably for the new size of the cross-section. This is attributed to the modification of damping parameter of the beam because of the change in the second moment of area of the beam cross-section. Consequently, changing the aspect ratio of the beam cross-section, while keeping the beam cross-sectional area same, modifies flexural and torsional oscillations of the beam, i.e., lowering aspect ratio (b/h) lowers the damping factor and the beam undergoes elastic like oscillations with high frequency.

$d_1 = d_2 = 0$	$d_1 \neq 0,\ d_2 = 0$	$d_1 \neq 0$, and $d_2 \neq 0$

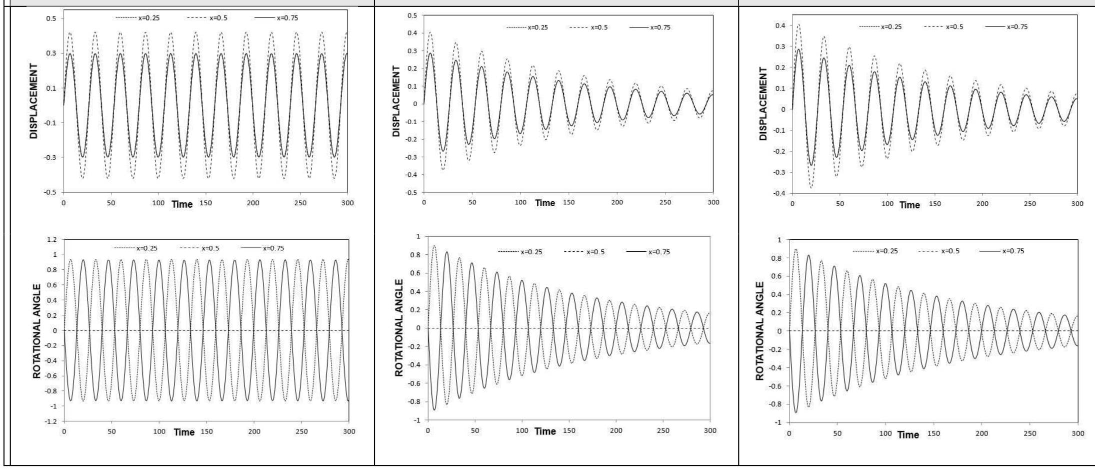


Figure 3.2: Flexural and rotational oscillations of hinged-hinged concrete beam for different damping conditions. The cross-sectional size of the rectangular beam is width (b) = 0.02 m and height (h) = 0.05 m. The length of the beam is 1 m. d1 (0.012) is the flexural damping factor and d2 (5.01×10^{-6}) is the torsional damping factor of concrete. x- axis locations represent the locations along the beam length.

Figure (3.3) shows flexural and torsional displacements of steel beam with the cross-sectional dimensions of $b = 1$ cm and $h = 10$ cm and length of 1 m for the cases: i) no flexural and torsional damping, ii) no torsional damping, but flexural damping is introduced, and iii) damping is introduced for both torsional and flexural motions. Elastic behavior is observed when there is no damping for both flexural and torsional oscillations and natural frequency is dominant for the flexural motion. For the case when the damping is introduced for the flexural motion only, the amplitude of the oscillation decays gradually with progressing time and the decay rate remains small in the amplitude of flexural oscillation. This argument is also true for the torsional oscillation, provided that the decay rate of the amplitude of the oscillation is significantly smaller than that of the flexural motion. In addition, the damping frequency is close to the natural frequency because of the small damping coefficient as compared to that of the concrete beam (figure (3.1)).The logarithmic decay of amplitude of oscillation increases when the damping is introduced for both flexural and torsional motions,. This is more pronounced for the torsional motion. However, the damping frequency differs slightly for torsional motion, which attains lower value than that of the flexural motion. Consequently, steel beam shows almost elastic behavior for the external excitation despite the damping is introduced, which is more pronounced for the flexural motion. When comparing figures (3.1) and (3.3), the logarithmic decay of the amplitude of the flexural and torsional oscillations is significantly higher for the concrete beam as compared to that of steel bar. The difference in the amplitude is because of the damping factor of the beam material; in which case, the viscous damping ratio of concrete is 0.017 while it is 0.002 for steel. Therefore, steel beam undergoes high cyclic fatigue as compare to that of the concrete beam, since the amplitude of both flexural and torsional oscillations decays gradually for steel.

$d_1 = d_2 = 0$	$d_1 \neq 0,\ d_2 = 0$	$d_1 \neq 0$, and $d_2 \neq 0$

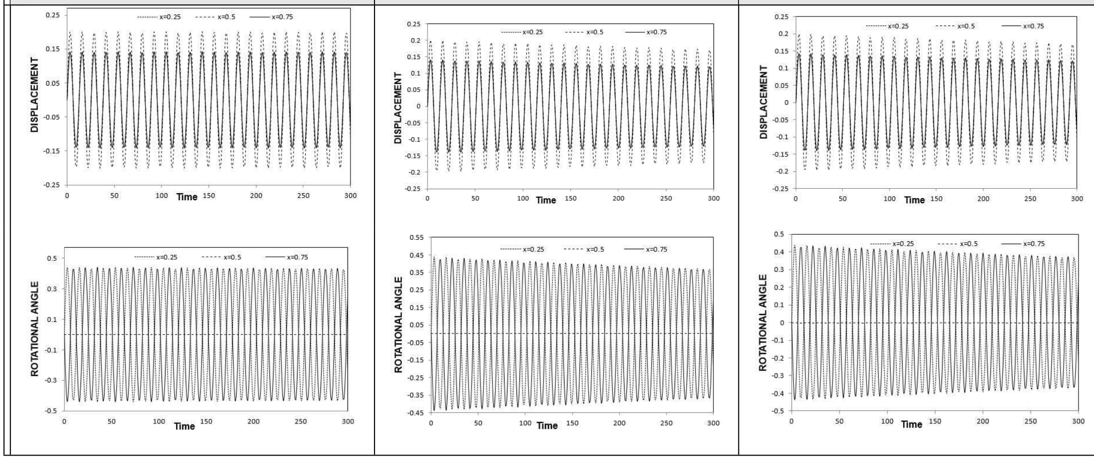


Figure 3.3: Flexural and rotational oscillations of hinged-hinged steel beam for different damping conditions. The cross-sectional size of the rectangular beam is width (b) = 0.01 m and height (h) = 0.1 m. The length of the beam is 1 m. The flexural damping factor is d1 (0.0011) and the torsional damping factor is d2 (3.55×10^{-7}) of steel beam. x- axis locations represent the locations along the beam length.

Figure (3.4) shows flexural and torsional oscillations of steel bar with the cross-sectional dimensions of $b = 2$ cm and $h = 5$ resulting the same cross-sectional area of that corresponding to figure (3.2). The characteristics of flexural and torsional oscillations change with changing the dimensions of cross-section (aspect ratio, b/h) of the beam. In this case, logarithmic decay of amplitude of flexural and torsional oscillations reduces significantly as similar to those shown in figure (3.2). This is again associated with the change of second moment of an area, i.e. reducing aspect ratio (b/h) lowers the damping factor and beam demonstrates elastic like flexural and torsional oscillations. Therefore, the occurrence of high cyclic fatigue is most likely for the hinged-hinged beam with low aspect ratios.

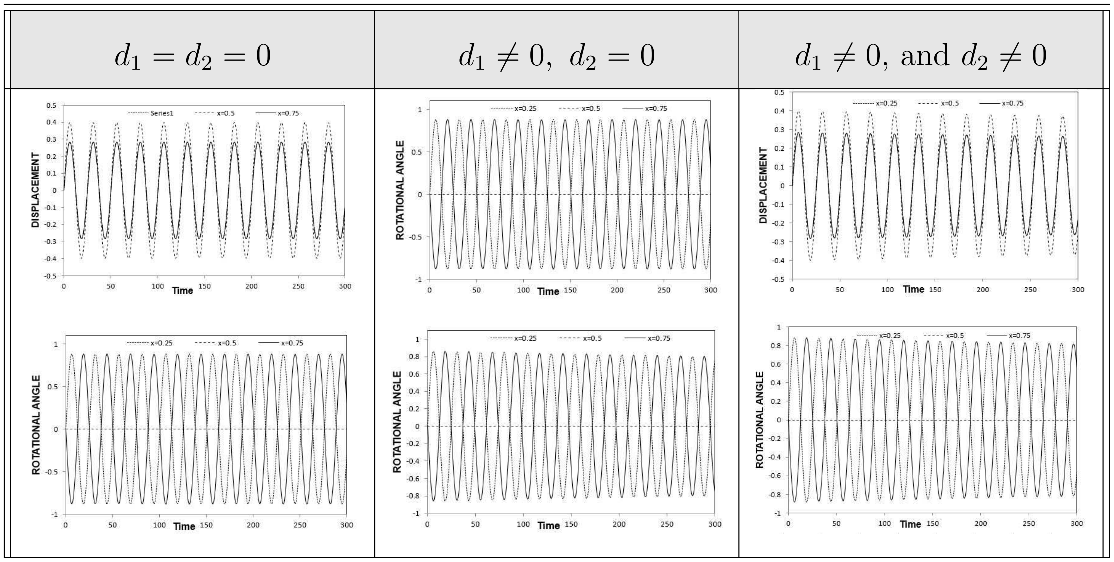


Figure 3.4: Flexural and rotational oscillations of hinged-hinged steel beam for different damping conditions. The cross-sectional size of the rectangular beam is width (b) = 0.02 m and height (h) = 0.05 m. The length of the beam is 1 m. The flexural damping factor is d1 (0.00053) and the torsional damping factor is d2 ($3.39 \times 10{-}7$) of steel beam. x- axis locations represent the locations along the beam length.

3.4 Conclusion

Analytical solution for the Timoshenko system with hinged-hinged configuration is presented. External excitation for flexural and torsional oscillations is introduced to assess the dynamic response of the beam. The dimension of the beam cross-section is varied with changing the aspect ratio of the cross-section (width and height are varied) while keeping the cross-sectional area constant. This arrangement enables to examine the effect of beam size on the flexural and torsional characteristics of the beam. Two beam materials are incorporated in the simulations, namely, concrete and steel. It is found that without introducing damping, flexural and torsional oscillations follow the elastic behavior; in which case, the natural frequency and the oscillation amplitude remain high, which is more pronounced for steel beam. Introducing damping in the flexural motion, while considering elastic behavior for torsional motion, results in logarithmic amplitude decay for flexural and torsional oscillations. The coupling of governing equations of motion is responsible for the amplitude decay in torsional oscillation despite the damping coefficient is set to zero for the torsional motion. The logarithmic decay of the amplitude in both flexural and torsional oscillations is more pronounced for concrete beam. This behavior is associated with the larger damping coefficient of concrete beam than that corresponding to steel beam. The damping frequency of the flexural and the torsional oscillations is also lower for concrete beam than that of steel beam. Consequently, steel beam undergoes high cyclic oscillation than concrete beam, which indicates that steel bar is more prone to the high cyclic fatigue than concrete beam, i.e. under same conditions of external excitation for flexural and torsional oscillations; steel beam suffers from high cyclic oscillation and resulting fatigue. The effect of size of the beam cross-section on the flexural and the torsional oscillations is significant for concrete and steel beams. In this case, the damping factor of beam changes due to the change of second moment of area, which is used for calculations of the damping factor, i.e. reducing damping factor, due to the change of beam cross-sectional size, gives rise to increased amplitude and frequency of the oscillation of the beam regardless of the beam material. The present study provides useful information on the dynamic response of hinged-hinged beam, resembling actual structures, when externally excited and enhances the understanding of the effect of beam size and material on the flexural and the torsional response of beam.

3.5 General Formula of Quartic Polynomial

The roots of any fourth order equation of the form

$$AX^4 + BX^3 + CX^2 + DX + E = 0,$$

can be found using the formula

$$X_{1,2} = \tfrac{-B}{4A} - S \pm \tfrac{1}{2}\sqrt{-4S^2 - 2p + \tfrac{q}{S}},$$
$$X_{3,4} = \tfrac{-B}{4A} + S \pm \tfrac{1}{2}\sqrt{-4S^2 - 2p - \tfrac{q}{S}},,$$

where

$$p = \tfrac{8AC-3B^2}{8A^2},\ \ q = \tfrac{B^3-4ABC+8A^2D}{8A^3},\ \ S = \tfrac{1}{2}\sqrt{\tfrac{-2}{3}p + \tfrac{1}{3A}\left(Q + \tfrac{\Delta_0}{Q}\right)},$$

with

$$Q = \sqrt[3]{\tfrac{\Delta_1+\sqrt{\Delta_1^2-4\Delta_0^3}}{2}},$$

such that

$$\Delta_0 = C^2 - 3BD + 12AE,\ \ \Delta_1 = 2C^3 - 9BCD + 27B^2E + 27AD^2 - 72ACE.$$

The discriminant Δ of the quartic equation can be computed using the formula

$$-27\Delta = \Delta_1^2 - 4\Delta_0^3 \, .$$

CHAPTER 4

DAMPED TIMOSHENKO BEAM WITH NON-LINEAR ROTATIONAL MOMENT

In this chapter we consider a non-linear Timoshenko system of PDEs with frictional damping term in rotation angle. The non-linearity is due to the arbitrary dependence on the rotation moment. We note that due to the nonlinearity, the Fourier transform approach can not be used. We therefore, adopt the Lie Symmetry analysis to study such problems. A Lie symmetry group classification of the arbitrary function of rotation moment is presented. Optimal system of one-dimensional subalgebras of the non-linear damped Timoshenko system are derived for all the non-linear cases. All possible invariant variables of the optimal systems for the three non-linear cases are presented. The corresponding reduced systems of ordinary differential equations ODEs are also provided.

4.1 Introduction

The classification of group invariant solutions of differential equations by means of the optimal systems is one of the main applications of Lie group analysis to differential equations. The method was first introduced by Ovsiannikov [53]. The main idea behind the method is discussed in his papers [54, 55] and also by Chupakin [13] and Ibragimov et al [37] and Olver [51]. We can always construct a family of group invariant solutions corresponding to a subgroup of a symmetry group admitted by a given differential equation. Since there are an infinite number of such subgroups, it is not possible to list all the group invariant solutions. An effective and systematic way of classifying these group invariant solutions is to obtain optimal systems of subalgebras of the symmetry Lie algebra. This leads to non-similar invariant solutions under symmetry transformations.

Timoshenko [80] proposed a beam theory which adds the effect of shear as well as the effect of rotation to the Euler-Bernoulli beam. The Timoshenko model is a major improvement for non-slender beams and for high-frequency response where shear or rotary effects are not negligible [28]. Rivera et. al. [63] studied the global stability for the following damped Timoshenko beam system with non-linear rotation moment

$$\begin{aligned} &\rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0, \\ &\rho_2\psi_{tt} - (\chi(\psi_x))_x + k(\varphi_x + \psi) + d\psi_t = 0. \end{aligned} \tag{4.1.1}$$

where the functions φ, ψ depending on $(t, x) \in (0, \infty) \times (0, L)$ model the transverse displacement of a beam with $(0, L) \in \Re$, and rotation angle of a filament, respectively. The constants ρ_1, ρ_2, d and k are positive and χ is a function of ψ_x assumed to satisfy

$$\chi_{\psi_x}(0) = b, \tag{4.1.2}$$

with positive constant b. However, the algebraic properties of the Lie algebra admitted by the (4.1.1) have not been studied so far. In this paper we perform a Lie symmetry analysis of non-linear damped Timoshenko beam system (4.1.1). In sections two, the complete Lie group classification is presented using Janet basis. In section three, optimal system of one-dimensional subalgebras of the non-linear damped Timoshenko system are derived for all the non-linear cases. In section four, all possible invariant variables of the optimal systems for the three non-linear cases of $\chi(\psi_x)$ are presented. Moreover, the corresponding reduced systems of ODEs are also provided. As an illustration, some invariant solutions are given explicitly in three examples by solving the reduced systems of ODEs.

4.2 Complete Lie group classification

Consider the system of two PDEs with two independent variables (t, x) and two dependent variables (φ, ψ) and the function $\chi = \chi(\psi_x)$ given by the system (4.1.1).

Consider the following symmetry transformation group acting on system of

PDEs (4.1.1).

$$\begin{aligned}\tilde{t} &= t+\epsilon\xi^1(t,x,\varphi,\psi)+O(\epsilon^2), \quad \tilde{x} = x+\epsilon\xi^2(t,x,\varphi,\psi)+O(\epsilon^2),\\ \tilde{\varphi} &= \varphi+\epsilon\eta^1(t,x,\varphi,\psi)+O(\epsilon^2), \quad \tilde{\psi} = \psi+\epsilon\eta^2(t,x,\varphi,\psi)+O(\epsilon^2),\end{aligned} \tag{4.2.1}$$

where ϵ is the group parameter and $\xi^i = \xi^i(t,x,\varphi,\psi)$ and $\eta^j = \eta^j(t,x,\varphi,\psi)$ for $i,j = 1,2$, are the infinitesimals of transformations for the independent and dependent variables, respectively. The associated Lie point symmetry generator (vector field) of the system (4.1.1) is of the form

$$X = \xi^1\tfrac{\partial}{\partial t}+\xi^2\tfrac{\partial}{\partial x}+\eta^1\tfrac{\partial}{\partial \varphi}+\eta^2\tfrac{\partial}{\partial \psi}. \tag{4.2.2}$$

The second prolongation of the generator is given by

$$X^{[2]} = X+\eta^{\mu}_{i_1}(x^j,u^j,\partial u^j)\frac{\partial}{\partial u^{\mu}_{i_1}}+\eta^{\mu}_{i_1i_2}(x^j,u^j,\partial u^j,\partial^2u^j)\frac{\partial}{\partial u^{\mu}_{i_1i_2}}, \tag{4.2.3}$$

such that $j=1,2$, and $(x^1,x^2)=(t,x)$, and $(u^1,u^2)=(\varphi,\psi)$, and $(\partial u^1,\partial u^2) = (\partial\varphi,\partial\psi)$, and so on... , where

$$\eta^{\mu}_{i_1} = D_{i_1}\eta^{\mu}-\sum_{j=1}^{2}(D_{i_1}\xi^j)u^{\mu}_j, \quad \mu=1,2\ ,$$

$$\eta^{\mu}_{i_1i_2} = D_{i_2}\eta^{\mu}_{i_1}-\sum_{j=1}^{2}(D_{i_2}\xi^j)u^{\mu}_{i_1j},$$

$$D_i = \frac{\partial}{\partial x_i}+u^{\mu}_i\frac{\partial}{\partial u^{\mu}}+u^{\mu}_{ij}\frac{\partial}{\partial u^{\mu}_j}+u^{\mu}_{ii_1i_2}\frac{\partial}{\partial u^{\mu}_{i_1i_2}}+...,\tag{4.2.4}$$

where D_i is the total derivative operator.

Using the invariance condition of the system of PDEs (4.1.1)

$$\begin{aligned} &X^{[2]}(\rho_1\varphi_{tt} - k(\varphi_x + \psi)_x)|_{(4.1.1)} = 0, \\ &X^{[2]}(\rho_2\psi_{tt} - \chi_x(\psi_x) + k(\varphi_x + \psi) + d\psi_t)|_{(4.1.1)} = 0, \end{aligned} \tag{4.2.5}$$

and comparing coefficients of the various derivatives of the dependent variables ϕ and ψ yields an over-determined linear PDE system. Carrying out the Janet basis of this over-determined system in the degree reverse lexicographical ordering as $\psi > \phi > x > t$ and $\eta_2 > \eta_1 > \xi_2 > \xi_1$ by using the command "JanetBasis" involved in the Maple package "Janet" [4], leads to two cases. These two cases arise from the command "Denominators" which returns the functions by which the Janet basis algorithm had to divide. These two cases are given as follows:

4.2.1 Linear rotational moment $\chi(\psi_x) = b\psi_x + \gamma$

In this case $\chi(\psi_x)$ is a linear function satisfying the condition (4.1.2). The Janet basis of of the over-determined system is

$$\begin{aligned} &[\ \eta^1_\psi,\ \xi^2_\psi,\ \xi^1_\psi,\ \eta^2_\phi,\ \eta^1_\phi - \eta^2_\psi,\ \xi^2_\phi,\ \xi^1_\phi,\ \xi^2_x,\ \xi^1_x,\ \xi^2_t,\ \xi^1_t,\ \eta^2_{\psi\psi},\ \eta^2_{x\psi},\ \eta^2_{t\psi},\ \eta^1_{t\psi},\ \eta^2_{x\phi},\ \eta^2_{t\phi}, \\ &\eta^1_{t\phi},\ \tfrac{k}{\rho 2}\eta^1_x + \tfrac{k}{\rho_2}\eta^2 + \tfrac{d}{\rho_2}\eta^2_t - \tfrac{k}{\rho_2}\eta^2_\psi\psi - \tfrac{b}{\rho_2}\eta^2_{x,x} + \eta^2_{t,t},\ -\tfrac{k}{\rho_1}\eta^1_{x,x} + \eta^1_{t,t} - \tfrac{k}{\rho_1}\eta^2_x\ ]. \end{aligned} \tag{4.2.6}$$

The solution of this system of determining equations is

$$\xi^1 = c_1, \quad \xi^2 = c_2, \quad \eta^1 = c_3\varphi + f(t,x), \quad \eta^2 = c_3\psi + g(t,x), \tag{4.2.7}$$

where $f(t,x)$ and $g(t,x)$ satisfy the following system of PDEs

$$\begin{aligned} &kf_x + kg + dg_t - bg_{xx} + \rho_2 g_{tt} = 0, \\ &\rho_1 f_{tt} - kf_{xx} - kg_x = 0. \end{aligned} \tag{4.2.8}$$

The corresponding Lie point symmetry generators admitted by the system (4.1.1) are given as

$$X_1 = \tfrac{\partial}{\partial t}, \quad X_2 = \tfrac{\partial}{\partial x}, \quad X_3 = \varphi\tfrac{\partial}{\partial \varphi} + \psi\tfrac{\partial}{\partial \psi}, \quad X_\infty = f(t,x)\tfrac{\partial}{\partial \varphi} + g(t,x)\tfrac{\partial}{\partial \psi}, \tag{4.2.9}$$

4.2.2 Non-Linear rotational moment $\chi_{\psi_x\psi_x} \neq 0$

The Janet basis of the over-determined system is

$$\begin{aligned} &[\ \eta^2_\psi,\ \eta^1_\psi,\ \xi^2_\psi,\ \xi^1_\psi,\ \eta^2_\phi,\ \eta^1_\phi,\ \xi^2_\phi,\ \xi^1_\phi,\ \eta^2_x,\ \xi^2_x,\ \xi^1_x,\ \xi^2_t,\ \xi^1_t, \eta^1_{x,\psi},\ \eta^2_{t,\psi}, \eta^1_{t,\psi},\ \eta^1_{x,\phi},\ \eta^2_{t,\phi}, \\ &\eta^1_{t,\phi},\ \eta^1_{x,x},\ \eta^2_{t,x},\ \tfrac{k}{\rho_2}\eta^1_x + \tfrac{k}{\rho_2}\eta^2 + \tfrac{d}{\rho_2}\eta^2_t + \eta^2_{t,t},\ \eta^1_{t,t},\ \eta^1_{t,x,\psi},\ \eta^1_{t,x,\phi},\ \eta^1_{t,x,x}\ ] \end{aligned} \tag{4.2.10}$$

The solution of the system (4.2.10) is

$$\xi^1 = c_1, \quad \xi^2 = c_2, \quad \eta^1 = c_3 + c_4\ t + c_5 x + c_6\ tx, \quad \eta^2 = F(t) \tag{4.2.11}$$

where $F(t)$ satisfies the following ODE

$$\rho_2 F''(t) + dF'(t) + kF(t) = -c_6 kt - c_5 k. \tag{4.2.12}$$

The characteristic equation of left hand side of the equation (4.2.12) gives rise to the following three sub-cases

Case1: $d^2 - 4k\rho_2 = 0$

The Lie point symmetry generators admitted by the system (4.1.1) are given by

$$\begin{aligned}
&X_1 = \tfrac{\partial}{\partial t} && X_2 = \tfrac{\partial}{\partial x}, && X_3 = \tfrac{\partial}{\partial \varphi}, \\
&X_4 = t\tfrac{\partial}{\partial \varphi}, && X_5 = x\tfrac{\partial}{\partial \varphi} - \tfrac{\partial}{\partial \psi}, && X_6 = tx\tfrac{\partial}{\partial \varphi} + (2\sqrt{\tfrac{\rho_2}{k}} - t)\tfrac{\partial}{\partial \psi}, \\
&X_7 = e^{-\sqrt{\frac{k}{\rho_2}}t}\tfrac{\partial}{\partial \psi}, && X_8 = te^{-\sqrt{\frac{k}{\rho_2}}t}\tfrac{\partial}{\partial \psi}.
\end{aligned} \tag{4.2.13}$$

In order to obtain the group transformations which are generated by the resulting infinitesimal symmetry generators (4.2.13), we need to solve the following system of first order ODEs

$$\begin{aligned}
&\tfrac{d\tilde{x}^j(\epsilon)}{d\epsilon} = \xi^j(\tilde{t}(\epsilon), \tilde{x}(\epsilon), \tilde{\varphi}(\epsilon), \tilde{\psi}(\epsilon)), \quad \tilde{x}^j(0) = x^j, \\
&\tfrac{d\tilde{u}^j(\epsilon)}{d\epsilon} = \eta^j(\tilde{t}(\epsilon), \tilde{x}(\epsilon), \tilde{\varphi}(\epsilon), \tilde{\psi}(\epsilon)), \quad \tilde{u}^j(0) = u^j, \quad j = 1, 2.
\end{aligned} \tag{4.2.14}$$

The one parameter group $G_i(\epsilon) = e^{\epsilon X_i}$ generated by X_i for $i = 1, ..., 8$, are as follows:

$$
\begin{aligned}
&G_1(\epsilon) : (t, x, \varphi, \psi) \mapsto (t + \epsilon, x, \varphi, \psi),\\
&G_2(\epsilon) : (t, x, \varphi, \psi) \mapsto (t, x + \epsilon, \varphi, \psi),\\
&G_3(\epsilon) : (t, x, \varphi, \psi) \mapsto (t, x, \varphi + \epsilon, \psi),\\
&G_4(\epsilon) : (t, x, \varphi, \psi) \mapsto (t, x, \varphi + \epsilon t, \psi),\\
&G_5(\epsilon) : (t, x, \varphi, \psi) \mapsto (t, x, \varphi + \epsilon x, \psi - \epsilon),\\
&G_6(\epsilon) : (t, x, \varphi, \psi) \mapsto (t, x, \varphi + \epsilon tx, \psi + \epsilon(2\sqrt{\tfrac{\rho_2}{k}} - t)),\\
&G_7(\epsilon) : (t, x, \varphi, \psi) \mapsto (t, x, \varphi, \psi + \epsilon e^{-\sqrt{\frac{k}{\rho_2}}t}),\\
&G_8(\epsilon) : (t, x, \varphi, \psi) \mapsto (t, x, \varphi, \psi + \epsilon t e^{-\sqrt{\frac{k}{\rho_2}}t}).
\end{aligned}
\tag{4.2.15}
$$

Theorem 4.2.1 *If $\varphi = f(t, x)$ and $\psi = g(t, x)$ is a solution of the Timoshenko system (4.1.1) with $d^2 - 4k\rho_2 = 0$, then so is*

$$
\begin{aligned}
\varphi &= f(t + \epsilon_1, x + \epsilon_2) + \epsilon_3 + \epsilon_4(t + \epsilon_1) + \epsilon_5(x + \epsilon_2) + \epsilon_6(t + \epsilon_1)(x + \epsilon_2),\\
\psi &= g(t + \epsilon_1, x + \epsilon_2) + 2\sqrt{\tfrac{\rho_2}{k}}\epsilon_6 - \epsilon_6(t + \epsilon_1) - \epsilon_5 + \epsilon_7 e^{-\sqrt{\frac{k}{\rho_2}}(t+\epsilon_1)}\\
&\quad + \epsilon_8(t + \epsilon_1)e^{-\sqrt{\frac{k}{\rho_2}}(t+\epsilon_1)},
\end{aligned}
\tag{4.2.16}
$$

where $\{\epsilon_i\}_{i=1}^{8}$ are arbitrary real numbers.

Proof. *The eight parameters group*

$$G(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6, \epsilon_7, \epsilon_8) = G_8(\epsilon_8) \circ G_7(\epsilon_7) \circ \ ... \ \circ G_1(\epsilon_1)$$

generated by X_i for $i = 1, ..., 8,$ can be given by the composition of the transformations (4.2.15) as follows:

$$\begin{aligned} G: \quad (t, x, \varphi, \psi) \longmapsto & (t+\epsilon_1,\ x+\epsilon_2,\ \varphi+\epsilon_3+\epsilon_4(t+\epsilon_1)+\epsilon_5(x+\epsilon_2), \\ & +\epsilon_6(t+\epsilon_1)(x+\epsilon_2),\ \psi+2\sqrt{\tfrac{\rho_2}{k}}\epsilon_6-\epsilon_6(t+\epsilon_1)-\epsilon_5. \\ & +e^{-\sqrt{\frac{k}{\rho_2}}(t+\epsilon_1)}(\epsilon_7+\epsilon_8(t+\epsilon_1))), \end{aligned} \tag{4.2.17}$$

and this completes the proof. ▮

Case2: $d^2 - 4k\rho_2 = \lambda^2$**, such that** $\lambda > 0$

The Lie point symmetry generators admitted by the system (4.1.1) are given by

$$\begin{aligned} X_1 &= \tfrac{\partial}{\partial t}, & X_2 &= \tfrac{\partial}{\partial x}, \\ X_3 &= \tfrac{\partial}{\partial \varphi}, & X_4 &= t\tfrac{\partial}{\partial \varphi}, \\ X_5 &= x\tfrac{\partial}{\partial \varphi} - \tfrac{\partial}{\partial \psi}, & X_6 &= tx\tfrac{\partial}{\partial \varphi} + (\tfrac{\sqrt{\lambda^2+4k\rho_2}}{k} - t)\tfrac{\partial}{\partial \psi}, \\ X_7 &= e^{-\frac{\sqrt{\lambda^2+4k\rho_2}}{2\rho_2}t}\cosh(\tfrac{\lambda t}{2\rho_2})\tfrac{\partial}{\partial \psi}, & X_8 &= e^{-\frac{\sqrt{\lambda^2+4k\rho_2}}{2\rho_2}t}\sinh(\tfrac{\lambda t}{2\rho_2})\tfrac{\partial}{\partial \psi}. \end{aligned} \tag{4.2.18}$$

The one parameter group $G_i(\epsilon) = e^{\epsilon X_i}$ generated by X_i for $i = 1, ..., 8$ are as follows: $G_i(\epsilon), i = 1, ..., 5$ are the same as in equation (4.2.15), and $G_i(\epsilon), i = 6, 7, 8$ are given by

$$\begin{aligned} G_6(\epsilon) &: (t, x, \varphi, \psi) \mapsto (t, x, \varphi+\epsilon tx, \psi+\epsilon(\tfrac{\sqrt{\lambda^2+4k\rho_2}}{k} - t), \\ G_7(\epsilon) &: (t, x, \varphi, \psi) \mapsto (t, x, \varphi, \psi+\epsilon e^{-\frac{\sqrt{\lambda^2+4k\rho_2}}{\rho_2}t}\cosh(\tfrac{\lambda t}{2\rho_2})), \\ G_8(\epsilon) &: (t, x, \varphi, \psi) \mapsto (t, x, \varphi, \psi+\epsilon e^{-\frac{\sqrt{\lambda^2+4k\rho_2}}{\rho_2}t}\sinh(\tfrac{\lambda t}{2\rho_2})). \end{aligned} \tag{4.2.19}$$

Theorem 4.2.2 *If $\varphi = f(t,x)$ and $\psi = g(t,x)$ is a solution of the Timoshenko system (4.1.1) with $d^2 - 4k\rho_2 = \lambda^2$, then so is*

$$\begin{aligned}\varphi = \ & F(t+\epsilon_1, x+\epsilon_2) + \epsilon_3 + \epsilon_4(t+\epsilon_1) + \epsilon_5(x+\epsilon c_2) + \epsilon_6(t+\epsilon_1)(x+\epsilon_2),\\ \psi = \ & G(t+\epsilon_1, x+\epsilon_2) - \epsilon_6(t+\epsilon_1) - \epsilon_5 + \tfrac{d}{k}\epsilon_6 + e^{-\frac{(dt+\epsilon_1)}{2\rho_2}}(\epsilon_7 \cosh(\tfrac{\lambda(t+\epsilon_1)}{2\rho_2})\\ & + \epsilon_8 \sinh(\tfrac{\lambda(t+\epsilon_1)}{2\rho_2})),\end{aligned} \tag{4.2.20}$$

where $\{\epsilon_i\}_{i=1}^{8}$ are arbitrary real numbers.

Proof. *The eight parameters group*

$$G(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6, \epsilon_7, \epsilon_8) = G_8(\epsilon_8) \circ G_7(\epsilon_7) \circ \ \ldots \ \circ G_1(\epsilon_1)$$

generated by X_i for $i = 1, ..., 8$, can be given by the composition of the transformations (4.2.19) as follows:

$$\begin{aligned}G: \ \ (t,x,\varphi,\psi) \longmapsto & (t+\epsilon_1, \ x+\epsilon_2, \ \varphi + \epsilon_3 + \epsilon_4(t+\epsilon_1) + \epsilon_5(x+\epsilon_2)\\ & +\epsilon_6(t+\epsilon_1)(x+\epsilon_2), \ \psi - \epsilon_6(t+\epsilon_1) - \epsilon_5 + \tfrac{d}{k}\epsilon_6 + e^{-\frac{(dt+\epsilon_1)}{2\rho_2}}(\epsilon_7 \cosh(\tfrac{\lambda(t+\epsilon_1)}{2\rho_2})\\ & +\epsilon_8 \sinh(\tfrac{2\lambda(t+\epsilon_1)}{2\rho_2}))).\end{aligned} \tag{4.2.21}$$

and this completes the proof. ▮

Case3: $d^2 - 4k\rho_2 = -\mu^2$**, such that** $\mu > 0$

The Lie point symmetry generators admitted by the system (4.1.1) are given by

$$\begin{aligned} &X_1 = \frac{\partial}{\partial t}, && X_2 = \frac{\partial}{\partial x}, \\ &X_3 = \frac{\partial}{\partial \varphi}, && X_4 = t\frac{\partial}{\partial \varphi}, \\ &X_5 = x\frac{\partial}{\partial \varphi} - \frac{\partial}{\partial \psi}, && X_6 = tx\frac{\partial}{\partial \varphi} + (\frac{\sqrt{4k\rho_2-\mu^2}}{k} - t)\frac{\partial}{\partial \psi}, \\ &X_7 = e^{-\frac{\sqrt{4k\rho_2-\mu^2}}{2\rho_2}t}\cos(\frac{\mu t}{2\rho_2})\frac{\partial}{\partial \psi}, && X_8 = e^{-\frac{\sqrt{4k\rho_2-\mu^2}}{2\rho_2}t}\sin(\frac{\mu t}{2\rho_2})\frac{\partial}{\partial \psi}. \end{aligned} \tag{4.2.22}$$

The one parameter group $G_i(\epsilon) = e^{\epsilon X_i}$ generated by X_i for $i = 1, ..., 8$ are as follows: $G_i(\epsilon), i = 1, ..., 5$ are the same as in equation (4.2.15), and $G_i(\epsilon), i = 6, 7, 8$ are given by

$$\begin{aligned} &G_6(\epsilon) : (t, x, \varphi, \psi) \mapsto (t, x, \varphi + \epsilon t x, \psi + \epsilon(\frac{d}{k} - t), \\ &G_7(\epsilon) : (t, x, \varphi, \psi) \mapsto (t, x, \varphi, \psi + \epsilon e^{-\frac{d}{\rho_2}t}\cos(\frac{\mu t}{2\rho_2})), \\ &G_8(\epsilon) : (t, x, \varphi, \psi) \mapsto (t, x, \varphi, \psi + \epsilon e^{-\frac{d}{\rho_2}t}\sin(\frac{\mu t}{2\rho_2})). \end{aligned} \tag{4.2.23}$$

Theorem 4.2.3 *If $\varphi = f(t, x)$ and $\psi = g(t, x)$ is a solution of the Timoshenko system (4.1.1) with $d^2 - 4k\rho_2 = -\mu^2$, then so is*

$$\begin{aligned} \varphi = \; & F(t + \epsilon_1, x + \epsilon_2) + \epsilon_3 + \epsilon_4(t + \epsilon_1) + \epsilon_5(x + \epsilon_2) + \epsilon_6(t + \epsilon_1)(x + \epsilon_2) \\ \psi = \; & G(t + \epsilon_1, x + \epsilon_2) - \epsilon_6(t + \epsilon_1) - \epsilon_5 + \frac{d}{k}\epsilon_6 + e^{-\frac{d(t+\epsilon_1)}{2\rho_2}}(\epsilon_7\cos(\frac{\mu(t+\epsilon_1)}{2\rho_2}) \\ & + \epsilon_8\sin(\frac{\mu(t+\epsilon_1)}{2\rho_2})). \end{aligned} \tag{4.2.24}$$

where $\{\epsilon_i\}_{i=1}^{8}$ are arbitrary real numbers.

Proof. *The eight parameter group*

$$G(\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4,\epsilon_5,\epsilon_6,\epsilon_7,\epsilon_8) = G_8(\epsilon_8)\circ G_7(\epsilon_7)\circ\ \ldots\ \circ G_1(\epsilon_1)$$

generated by X_i for $i = 1, ..., 8$, can be given by the composition of the transformations (4.2.23) as follows:

$$\begin{aligned} G:\ \ &(t,x,\varphi,\psi)\longmapsto (t+\epsilon_1,\ x+\epsilon_2,\ \phi+\epsilon_3+\epsilon_4(t+\epsilon_1)+\epsilon_5(x+\epsilon_2)\\ &+\epsilon_6(t+\epsilon_1)(x+\epsilon_2),\ \psi-\epsilon_6(t+\epsilon_1)-\epsilon_5+\tfrac{d}{k}\epsilon_6+e^{-\frac{d(t+\epsilon_1)}{2\rho_2}}(\epsilon_7\cos(\tfrac{\mu(t+\epsilon_1)}{2\rho_2})\\ &+\ \epsilon_8\sin(\tfrac{\mu(t+\epsilon_1)}{2\rho_2})). \end{aligned} \tag{4.2.25}$$

and this completes the proof. ∎

4.3 One-dimensional optimal system

In this section, we give the complete classification of the one-dimensional optimal system for each of the algebras with basis (4.2.13), (4.2.18) and (4.2.22). In order to find the optimal system, one needs to classify the one-dimensional subalgebras under the action of the adjoint representation. We follow the algorithm explained by Olver [51].

First, we calculate the adjoint representation given by

$$Ad(exp(\epsilon X_i).X_j) = X_j - \epsilon[X_i,X_j] + \tfrac{\epsilon^2}{2!}[X_i,[X_i,X_j]] - \tfrac{\epsilon^3}{3!}[X_i,[X_i,X_j]] + ...,$$

and then establish the adjoint table. For each case, we classify the conjugacy classes under the adjoint representation according to the sign of the killing form.

4.3.1 Optimal system for the case $d^2 - 4k\rho_2 = 0$

The non-zero commutators of the Lie algebra $\mathcal{L}^8$ with basis (4.2.13) are given by

$$\begin{aligned} &[X_1, X_4] = X_3, && [X_1, X_6] = X_5\ , && [X_1, X_7] = -\sqrt{\tfrac{k}{\rho_2}}X_7, \\ &[X_1, X_8] = X_7 - \sqrt{\tfrac{k}{\rho_2}}X_8, && [X_2, X_5] = X_3, && [X_2, X_6] = X_4. \end{aligned} \tag{4.3.1}$$

The Lie algebra $\mathcal{L}^8$ is solvable and the Killing form is given by $K = 2\ \hat{a}^2 a_1^2$ where $\hat{a} = \sqrt{\frac{k}{\rho_2}}$. The adjoint table is given by

Table 4.1: The adjoint table corresponding to commutators (4.3.1)

$Ad(e^{\epsilon})$	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
X_1	X_1	X_2	X_3	$X_4 - \epsilon X_3$	X_5	$X_6 - \epsilon X_5$	$e^{\epsilon\hat{a}} X_7$	$e^{\epsilon\hat{a}} X_8 - \epsilon e^{\epsilon\hat{a}} X_7$
X_2	X_1	X_2	X_3	X_4	$X_5 - \epsilon X_3$	$X_6 - \epsilon X_4$	X_7	X_8
X_3	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
X_4	$X_1 + \epsilon X_3$	X_2	X_3	X_4	X_5	X_6	X_7	X_8
X_5	X_1	$X_2 + \epsilon X_3$	X_3	X_4	X_5	X_6	X_7	X_8
X_6	$X_1 + \epsilon X_5$	$X_2 + \epsilon X_4$	X_3	X_4	X_5	X_6	X_7	X_8
X_7	$X_1 - \hat{a}\epsilon X_7$	X_2	X_3	X_4	X_5	X_6	X_7	X_8
X_8	$X_1 + \epsilon X_7 - \hat{a}\epsilon X_8$	X_2	X_3	X_4	X_5	X_6	X_7	X_8

The adjoint group is defined by the matrix

$$A = Ad(e^{-\epsilon_8 X_8}).Ad(e^{-\epsilon_7 X_7}).\ \ldots\ .Ad(e^{-\epsilon_1 X_1}),$$

which is given by

$$A = \begin{pmatrix} 1 & 0 & -\epsilon_4 & 0 & -\epsilon_6 & 0 & \hat{a}\epsilon_7 - \epsilon_8 & \hat{a}\epsilon_8 \\ 0 & 1 & -\epsilon_5 & -\epsilon_6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \epsilon_1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \epsilon_2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \epsilon_1\epsilon_2 & \epsilon_2 & \epsilon_1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{-\hat{a}\epsilon_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \epsilon_1 e^{-\hat{a}\epsilon_1} & e^{-\hat{a}\epsilon_1} \end{pmatrix}. \tag{4.3.2}$$

Theorem 4.3.1 *An optimal system of one-dimensional Lie algebra $\mathcal{L}^8$ with basis (4.2.13) is provided by the following generators*

$$\begin{array}{ll|ll} X^1 = X_1 + \alpha X_2 + \beta X_6, & \alpha \in \mathbb{R}, \beta \neq 0, & X^8 = \alpha X_4 + X_5 + \beta X_8, & \alpha \in \mathbb{R}, \beta \neq 0, \\ X^2 = X_1 + \alpha X_2 + \beta X_4, & \alpha, \beta \in \mathbb{R}, & X^9 = \alpha X_4 + \beta X_5 + X_7, & \alpha \in \mathbb{R}, \beta \neq 0, \\ X^3 = X_2 + \alpha X_6 + \beta X_7 + \gamma X_8, & \alpha \neq 0, \beta, \gamma \in \mathbb{R}, & X^{10} = \alpha X_4 + X_5, & \alpha \in \mathbb{R}, \\ X^4 = X_2 + \alpha X_5 + \beta X_8, & \alpha \in \mathbb{R}, \beta \neq 0, & X^{11} = X_4 + \alpha X_7 + \beta X_8, & \alpha, \beta \in \mathbb{R}, \\ X^5 = \alpha X_2 + \beta X_5 + X_7, & \alpha \neq 0, \beta \in \mathbb{R}, & X^{12} = \alpha X_3 + X_8, & \alpha \in \mathbb{R}, \\ X^6 = X_2 + \alpha X_5, & \alpha \in \mathbb{R}, & X^{13} = \alpha X_3 + X_7, & \alpha \in \mathbb{R}, \\ X^7 = \alpha X_3 + X_6 + \beta X_7 + \gamma X_8, & \alpha, \beta, \gamma \in \mathbb{R}, & X^{14} = X_3. & \end{array} \tag{4.3.3}$$

Proof. *Let X and $\tilde{X}$ be two elements in the Lie algebra $\mathcal{L}^8$ with basis (3.2.18) given by $X = \sum_{i=1}^{8} a_i X_i$ and $\tilde{X} = \sum_{i=1}^{8} \tilde{a}_i X_i$. For simplicity, we will write X and $\tilde{X}$ as row vectors of the coefficients on the form $X = (\, a_1\ a_2\ \ldots\ a_8\,)$ and $\tilde{X} = (\, \tilde{a}_1\ \tilde{a}_2\ \ldots\ \tilde{a}_8\,)$. Then in order for X and $\tilde{X}$ to be in the same conjugacy* *class, we must have $\tilde{X} = XA$, where A is given by (4.3.2). So, the theorem is proved by solving the system*

$$
\begin{aligned}
&\tilde{a_1} = a_1,\\
&\tilde{a_2} = a_2,\\
&\tilde{a_3} = a_3 + \epsilon_1 a_4 + \epsilon_2 a_5 - \epsilon_5 a_2 - \epsilon_4 a_1 + \epsilon_1 \epsilon_2 a_6,\\
&\tilde{a_4} = a_4 - \epsilon_6 a_2 + \epsilon_2 a_6,\\
&\tilde{a_5} = a_5 - \epsilon_6 a_1 + \epsilon_1 a_6,\\
&\tilde{a_6} = a_6,\\
&\tilde{a_7} = a_1(\hat{a}\epsilon_7 - \epsilon_8) + e^{-\epsilon_1 \hat{a}}(a_7 + a_8 \epsilon_1),\\
&\tilde{a_8} = a_1 \hat{a} \epsilon_8 + a_8 e^{-\epsilon_1 \hat{a}},
\end{aligned}
\tag{4.3.4}
$$

for $\{\epsilon_i\}_{i=1}^{8}$ in term of $\{a_i\}_{i=1}^{8}$ in order to get the simplest values of $\{\tilde{a}_i\}_{i=1}^{8}$. The results are presented for different cases in the tree diagram 4.1 bellow, where it is initiated by the sign of the Killing form and its leafs are given completely.

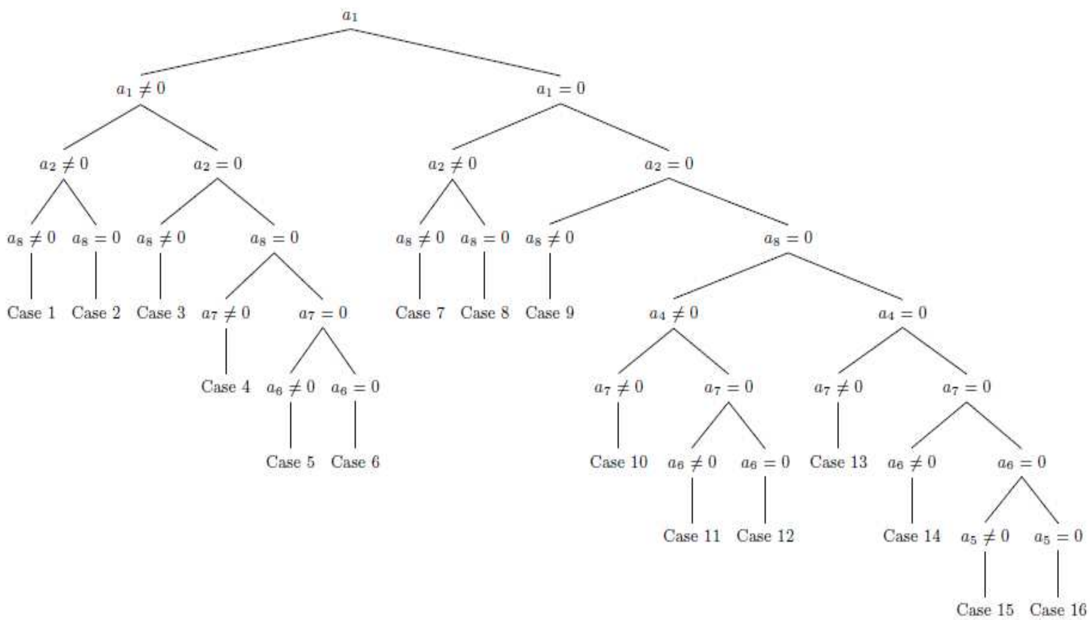


Figure 4.1: Tree diagram

The full details for each leaf are given as follows:

Case1: $a_1 \neq 0$, $a_6 \neq 0$ *: Let* $\epsilon_1 = \epsilon_5 = 0$, $\epsilon_2 = \frac{a_2a_5 - a_1a_4}{a_1a_6}$, $\epsilon_4 = \frac{a_1a_3a_6 - a_1a_4a_5 - a_2a_5}{a_1^2a_6}$, $\epsilon_6 = \frac{a_5}{a_1}$, $\epsilon_7 = -\frac{\gamma a_7 + a_8}{\gamma^2 a_1^2}$ *and* $\epsilon_8 = -\frac{a_8}{\gamma a_1}$ *to have* $\tilde{a}_3 = \tilde{a}_4 = \tilde{a}_5 = \tilde{a}_7 = \tilde{a}_8 = 0$*. Then the conjugacy class is* $\langle X_1 + \alpha X_2 + \beta X_6 \rangle$*, with* $\alpha \in \mathbb{R}, \beta \neq 0$*.*

Case2: $a_1 \neq 0$, $a_6 = 0$ *: Let* $\epsilon_1 = \epsilon_2 = \epsilon_5 = 0$, $\epsilon_4 = \frac{a_3}{a_1}$, $\epsilon_6 = \frac{a_5}{a_1}$, $\epsilon_7 = -\frac{\gamma a_7 + a_8}{\gamma^2 a_1}$, $\epsilon_8 = -\frac{a_8}{\gamma a_1}$ *to have* $\tilde{a}_3 = \tilde{a}_5 = \tilde{a}_7 = \tilde{a}_8 = 0$*. So the conjugacy class is* $\langle X_1 + \alpha X_2 + \beta X_4 \rangle$*, with* $\alpha, \beta \in \mathbb{R}$*.*

Case3: $a_1 = 0$, $a_2 \neq 0$, $a_6 \neq 0$ *: Let* $\epsilon_1 = -\frac{a_5}{a_6}$, $\epsilon_2 = 0$, $\epsilon_5 = \frac{a_3a_6 - a_4a_5}{a_2a_6}$*, and* $\epsilon_6 = \frac{a_4}{a_2}$ *to make* $\tilde{a}_3 = \tilde{a}_4 = \tilde{a}_5 = 0$*. Then the conjugacy class is* $\langle X_2 + \alpha X_6 + \beta X_7 + \gamma X_8 \rangle$, $\alpha \neq 0, \beta, \gamma \in \mathbb{R}$*.*

Case4: $a_1 = 0$, $a_2 \neq 0$, $a_6 = 0$, $a_8 \neq 0$ *: Let* $\epsilon_1 = -\frac{a_7}{a_8}$, $\epsilon_2 = 0$, $\epsilon_5 = \frac{a_3a_8 - a_4a_7}{a_2a_8}$*, and* $\epsilon_6 = \frac{a_4}{a_2}$ *to make* $\tilde{a}_3 = \tilde{a}_4 = \tilde{a}_7 = 0$*. Then the conjugacy class is of the form* $\langle X_2 + \alpha X_5 + \beta X_8 \rangle$, $\alpha \in \mathbb{R}, \beta \neq 0$*.*

Case5: $a_1 = 0$, $a_2 \neq 0$, $a_6 = 0$, $a_8 = 0$, $a_7 \neq 0$: Let $\epsilon_1 = \frac{\ln|a_7|}{\gamma}$, $\epsilon_2 = 0$, $\epsilon_5 = \frac{\frac{a_4}{\gamma}\ln|a_7|+a_7}{a_2}$ *and* $\epsilon_6 = \frac{a_4}{a_2}$ *to make* $\tilde{a_3} = \tilde{a_4} = 0$ *and* $\tilde{a_7} = \pm 1$. *Then the conjugacy class after consider appropriate scaling is* $\langle \alpha X_2 + \beta X_5 + X_7 \rangle$, *where* $\alpha \neq 0, \beta \in \mathbb{R}$.

Case6: $a_1 = 0$, $a_2 \neq 0$, $a_6 = 0$, $a_8 = 0$, $a_7 = 0$: *Let* $\epsilon_1 = \epsilon_2 = 0$ $\epsilon_5 = \frac{a_3}{a_2}$ *and* $\epsilon_6 = \frac{a_4}{a_2}$ *to make* $\tilde{a_3} = \tilde{a_4} = 0$. *Then the conjugacy class is* $\langle X_2 + \alpha X_5 \rangle$, $\alpha \in \mathbb{R}$.

Case7: $a_1 = 0$, $a_2 = 0$, $a_6 \neq 0$: *Let* $\epsilon_1 = -\frac{a_5}{a_6}$, $\epsilon_2 = -\frac{a_4}{a_6}$ *to make* $\tilde{a_4} = \tilde{a_5} = 0$. *Then the conjugacy class is of the form* $\langle \alpha X_3 + X_6 + \beta X_7 + \gamma X_8 \rangle$, $\alpha, \beta, \gamma \in \mathbb{R}$.

Case8: $a_1 = 0$, $a_2 = 0$, $a_6 = 0$, $a_5 \neq 0$, $a_8 \neq 0$: *Let* $\epsilon_1 = -\frac{a_7}{a_8}$ *and* $\epsilon_2 = \frac{a_4 a_7 - a_3 a_8}{a_5 a_8}$, *to get* $\tilde{a_3} = \tilde{a_7} = 0$. *Then the conjugacy class is of the form* $\langle \alpha X_4 + X_5 + \beta X_8 \rangle$, $\alpha \in \mathbb{R}, \beta \neq 0$.

Case9: $a_1 = 0$, $a_2 = 0$, $a_6 = 0$, $a_5 \neq 0$, $a_8 = 0$, $a_7 \neq 0$: *Let* $\epsilon_1 = \frac{\ln|a_7|}{\gamma}$ *with* $\epsilon_2 = -\frac{a_4}{\gamma a_5}\ln|a_7| - \frac{a_3}{a_7}$ *to make* $\tilde{a_3} = 0$, $\tilde{a_7} = \pm 1$. *Then conjugacy class after appropriate scaling is of the form* $\langle \alpha X_4 + \beta X_5 + X_7 \rangle$, $\alpha \in \mathbb{R}, \beta \neq 0$.

Case10: $a_1 = 0$, $a_2 = 0$, $a_6 = 0$, $a_5 \neq 0$, $a_8 = 0$, $a_7 = 0$: *Let* $\epsilon_2 = -\frac{a_3}{a_5}$ *to make* $\tilde{a_3} = 0$ *and so we have the conjugacy class of the form* $\langle \alpha X_4 + X_5 \rangle$, *with* $\alpha \in \mathbb{R}$.

Case11: $a_1 = 0$, $a_2 = 0$, $a_6 = 0$, $a_5 = 0$, $a_4 \neq 0$: *Let* $\epsilon_1 = -\frac{a_3}{a_4}$ *to make* $\tilde{a_3} = 0$, *Then the conjugacy class is* $\langle X_4 + \alpha X_7 + \beta X_8 \rangle$, $\alpha, \beta \in \mathbb{R}$.

Case12: $a_1 = 0$, $a_2 = 0$, $a_6 = 0$, $a_5 = 0$, $a_4 = 0$, $a_8 \neq 0$: *Let* $\epsilon_1 = -\frac{a_7}{a_8}$ *to have* $\tilde{a_7} = 0$. *Then the conjugacy class is* $\langle \alpha X_3 + X_8 \rangle$, $\alpha \in \mathbb{R}$.

Case13: $a_1 = 0$, $a_2 = 0$, $a_6 = 0$, $a_5 = 0$, $a_4 = 0$, $a_8 = 0$, $a_7 \neq 0$: *Let* $\epsilon_1 = \frac{\ln|a_7|}{\gamma}$ *to have* $\tilde{a_7} = \pm 1$ *and so the conjugacy class after appropriate scaling is* $\langle \alpha X_3 + X_7 \rangle$,

$\alpha \in \mathbb{R}$.

Case14: *$a_1 = 0,\ a_2 = 0,\ a_6 = 0,\ a_5 = 0,\ a_4 = 0,\ a_8 = 0,\ a_7 = 0$: Then directly we get a conjugacy class of the form $\langle X_3 \rangle$.*

∎

4.3.2 Optimal system for the case $d^2 - 4k\rho_2 = \lambda^2$

The non-zero commutators of the Lie algebra $\mathcal{L}^8$ with basis (4.2.18) are given by

$$\begin{aligned} &[X_1, X_4] = X_3, && [X_1, X_6] = X_5\ , && [X_1, X_7] = -\tfrac{d}{2\rho_2} X_7 + \tfrac{\lambda}{2\rho_2} X_8, \\ &[X_1, X_8] = -\tfrac{d}{2\rho_2} X_8 + \tfrac{\lambda}{2\rho_2} X_7, && [X_2, X_5] = X_3, && [X_2, X_6] = X_4. \end{aligned} \tag{4.3.5}$$

The Lie algebra $\mathcal{L}^8$ is solvable and the Killing form is given by $K = 2(\hat{a}^2 + \hat{b}^2)a_1^2$ where $\hat{a} = \frac{d}{2\rho_2}$ and $\hat{b} = \frac{\lambda}{2\rho_2}$. The adjoint table is given by

Table 4.2: The adjoint table corresponding to commutators (4.3.5)

$Ad(e^{\epsilon})$	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
X_1	X_1	X_2	X_3	$X_4 - \epsilon X_3$	X_5	$X_6 - \epsilon X_5$	Y_1	Y_2
X_2	X_1	X_2	X_3	X_4	$X_5 - \epsilon X_3$	$X_6 - \epsilon X_4$	X_7	X_8
X_3	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
X_4	$X_1 + \epsilon X_3$	X_2	X_3	X_4	X_5	X_6	X_7	X_8
X_5	X_1	$X_2 + \epsilon X_3$	X_3	X_4	X_5	X_6	X_7	X_8
X_6	$X_1 + \epsilon X_5$	$X_2 + \epsilon X_4$	X_3	X_4	X_5	X_6	X_7	X_8
X_7	$X_1 - \epsilon\hat{a}X_7 + \epsilon\hat{b}X_8$	X_2	X_3	X_4	X_5	X_6	X_7	X_8
X_8	$X_1 + \epsilon\hat{b}X_7 - \epsilon\hat{a}X_8$	X_2	X_3	X_4	X_5	X_6	X_7	X_8

$Y_1 = \frac{1}{2}(e^{\epsilon(\hat{a}+\hat{b})} + e^{\epsilon(\hat{a}-\hat{b})})X_7 + \frac{1}{2}(e^{\epsilon(\hat{a}+\hat{b})} - e^{\epsilon(\hat{a}-\hat{b})})X_8$

and $Y_2 = \frac{1}{2}(e^{\epsilon(\hat{a}+\hat{b})} - e^{\epsilon(\hat{a}-\hat{b})})X_7 + \frac{1}{2}(e^{\epsilon(\hat{a}+\hat{b})} + e^{\epsilon(\hat{a}-\hat{b})})X_8$.

The adjoint group is defined by the matrix

$$A = Ad(e^{-\epsilon_8 X_8}).Ad(e^{-\epsilon_7 X_7}).Ad(e^{-\epsilon_6 X_6}).Ad(e^{-\epsilon_5 X_5}).\ ...\ .Ad(e^{-\epsilon_1 X_1}),$$

which is given by

$$A = \begin{pmatrix} 1 & 0 & -\epsilon_4 & 0 & -\epsilon_6 & 0 & \hat{a}\epsilon_7 - \hat{b}\epsilon_8 & \hat{a}\epsilon_8 - \hat{b}\epsilon_7 \\ 0 & 1 & -\epsilon_5 & -\epsilon_6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \epsilon_1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \epsilon_2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \epsilon_1\epsilon_2 & \epsilon_2 & \epsilon_1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \hat{Y}_1 & \hat{Y}_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & \hat{Y}_2 & \hat{Y}_1 \end{pmatrix} \tag{4.3.6}$$

$\hat{Y}_1 = \frac{1}{2}(e^{-\epsilon_1(\hat{a}-\hat{b})} + e^{-\epsilon_1(\hat{a}+\hat{b})})$ and $\hat{Y}_2 = \frac{1}{2}(e^{-\epsilon_1(\hat{a}-\hat{b})} - e^{-\epsilon_1(\hat{a}+\hat{b})})$.

Theorem 4.3.2 *An optimal system of one-dimensional Lie algebra of L^8 with basis (4.2.18) is provided by the following generators*

$$
\begin{array}{llll}
X^1 = X_1 + \alpha X_2 + \beta X_6, & \alpha \in \mathbb{R}, \beta \neq 0, & X^{11} = \alpha X_4 + X_5 + \beta X_8, & \alpha \in \mathbb{R}, \beta \neq 0, \\
X^2 = X_1 + \alpha X_2 + \beta X_4, & \alpha, \beta \in \mathbb{R}, & X^{12} = \alpha X_4 + \beta X_5 + X_7 + X_8, & \alpha \in \mathbb{R}, \beta \neq 0, \\
X^3 = X_2 + \alpha X_6 + \beta X_7 + \gamma X_8, & \alpha \neq 0, \beta, \gamma \in \mathbb{R}, & X^{13} = \alpha X_4 + \beta X_5 + X_7 - X_8, & \alpha \in \mathbb{R}, \beta \neq 0, \\
X^4 = X_2 + \alpha X_5 + \beta X_7, & \alpha \in \mathbb{R}, \beta \neq 0, & X^{14} = \alpha X_4 + X_5, & \alpha \in \mathbb{R}, \\
X^5 = X_2 + \alpha X_5 + \beta X_8, & \alpha \in \mathbb{R}, \beta \neq 0, & X^{15} = X_4 + \alpha X_7 + \beta X_8, & \alpha, \beta \in \mathbb{R}, \\
X^6 = \alpha X_2 + \beta X_5 + X_7 + X_8, & \alpha \neq 0, \beta \in \mathbb{R}, & X^{16} = \alpha X_3 + X_7, & \alpha \in \mathbb{R}, \\
X^7 = \alpha X_2 + \beta X_5 + X_7 - X_8, & \alpha \neq 0, \beta \in \mathbb{R}, & X^{17} = \alpha X_3 + X_8, & \alpha \in \mathbb{R}, \\
X^8 = X_2 + \alpha X_5, & \alpha \in \mathbb{R}, & X^{18} = \alpha X_3 + X_7 + X_8, & \alpha \in \mathbb{R}, \\
X^9 = \alpha X_3 + X_6 + \beta X_7 + \gamma X_8, & \alpha, \beta, \gamma \in \mathbb{R}, & X^{19} = \alpha X_3 + X_7 - X_8, & \alpha \in \mathbb{R}, \\
X^{10} = \alpha X_4 + X_5 + \beta X_7, & \alpha \in \mathbb{R}, \beta \neq 0, & X^{20} = X_3. &
\end{array}
\tag{4.3.7}
$$

Proof. *Let X and $\tilde{X}$ be two elements in the Lie algebra $\mathcal{L}^8$ with basis (3.3) given by $X = \sum_{i=1}^{8} a_i X_i$ and $\tilde{X} = \sum_{i=1}^{8} \tilde{a}_i X_i$. For simplicity, we will write X and $\tilde{X}$ as row vectors of the coefficients on the form $X = (\, a_1 \; a_2 \; . \; . \; . \; a_8 \,)$ and $\tilde{X} = (\, \tilde{a_1} \; \tilde{a_2} \; . \; . \; . \; \tilde{a_8} \,)$.*

Then in order for X and $\tilde{X}$ to be in the same conjugacy class, we must have $\tilde{X} = XA$, where A is given by (4.3.6). So, the theorem is proved by solving the system

$$\tilde{a_1} = a_1,$$

$$\tilde{a_2} = a_2,$$

$$\tilde{a_3} = a_3 + \epsilon_1 a_4 + \epsilon_2 a_5 - \epsilon_5 a_2 - \epsilon_4 a_1 + \epsilon_1\epsilon_2 a_6,$$

$$\tilde{a_4} = a_4 - \epsilon_6 a_2 + \epsilon_2 a_6,$$

$$\tilde{a_5} = a_5 - \epsilon_6 a_1 + \epsilon_1 a_6,$$

$$\tilde{a_6} = a_6,$$

$$\tilde{a_7} = a_1(\hat{a}\epsilon_7 - \hat{b}\epsilon_8) + \tfrac{1}{2}a_7(e^{-\epsilon_1(\hat{a}-\hat{b})} + e^{-\epsilon_1(\hat{a}+\hat{b})}) + \tfrac{1}{2}a_8(e^{-\epsilon_1(\hat{a}-\hat{b})} - e^{-\epsilon_1(\hat{a}+\hat{b})}),$$

$$\tilde{a_8} = a_1(\hat{a}\epsilon_8 - \hat{b}\epsilon_7) + \tfrac{1}{2}a_7(e^{-\epsilon_1(\hat{a}-\hat{b})} - e^{-\epsilon_1(\hat{a}+\hat{b})}) + \tfrac{1}{2}a_8(e^{-\epsilon_1(\hat{a}-\hat{b})} + e^{-\epsilon_1(\hat{a}+\hat{b})}), \tag{4.3.8}$$

for $\{\epsilon_i\}_{i=1}^8$ in term of $\{a_i\}_{i=1}^8$ in order to get the simplest values of $\{\tilde{a}_i\}_{i=1}^8$. The results are presented for different cases in the tree diagram (2) given in the appendix where it is initiated by the sign of the Killing form and its leafs are given completely. The full details for each leaf are gives as follows:

Case1: $a_1 \neq 0$, $a_6 \neq 0$ *: Let* $\epsilon_1 = 0$, $\epsilon_2 = \frac{a_2a_5 - a_1a_4}{a_1a_6}$, $\epsilon_4 = \frac{a_1a_3a_6 - a_1a_4a_5 + a_2a_5^2}{a_1^2a_6}$, $\epsilon_5 = 0$, $\epsilon_6 = \frac{a_5}{a_1}$, $\epsilon_7 = -\frac{\hat{a}a_7 + \hat{b}a_8}{(\hat{a}-\hat{b})a_1}$ *and* $\epsilon_8 = -\frac{\hat{b}a_7 + \hat{a}a_8}{(\hat{a}-\hat{b})a_1}$ *to have* $\tilde{a}_3 = \tilde{a}_4 = \tilde{a}_5 = \tilde{a}_7 = \tilde{a}_8 = 0$, *then we obtain the conjugacy class* $\langle X_1 + \alpha X_2 + \beta X_6\rangle$, *where* $\alpha \in \mathbb{R}$, $\beta \neq 0$.

Case2: $a_1 \neq 0$, $a_6 = 0$ *: Let* $\epsilon_1 = \epsilon_2 = 0$, $\epsilon_4 = \frac{a_3}{a_1}$, $\epsilon_5 = 0$, $\epsilon_6 = \frac{a_5}{a_1}$, $\epsilon_7 = -\frac{\hat{a}a_7 + \hat{b}a_8}{(\hat{a}-\hat{b})a_1}$ *and* $\epsilon_8 = -\frac{\hat{b}a_7 + \hat{a}a_8}{(\hat{a}-\hat{b})a_1}$ *to have* $\tilde{a}_3 = \tilde{a}_5 = \tilde{a}_6 = \tilde{a}_7 = \tilde{a}_8 = 0$ *and so we obtain the donjugacy class* $\langle X_1 + \alpha X_2 + \beta X_4\rangle$, *where* $\alpha, \beta \in \mathbb{R}$.

Case3: $a_1 = 0$, $a_2 \neq 0$, $a_6 \neq 0$ *: Let* $\epsilon_1 = -\frac{a_5}{a_6}$, $\epsilon_2 = 0$, $\epsilon_5 = \frac{a_3a_6 - a_4a_5}{a_2a_6}$ *and* $\epsilon_6 = \frac{a_4}{a_2}$ *to have* $\tilde{a}_3 = \tilde{a}_4 = \tilde{a}_5 = 0$. *Then the conjugacy class is of the form* $\langle X_2 + \alpha X_6 + \beta X_7 + \gamma X_8\rangle$, *where* $\alpha \neq 0$, $\beta, \gamma \in \mathbb{R}$.

Case4: *$a_1 = 0$, $a_2 \neq 0$, $a_6 = 0$, $a_7 + a_8 \neq 0$, $\frac{a_7-a_8}{a_7+a_8} > 0$: Let $\epsilon_1 = \frac{1}{2\hat{b}} \ln(\frac{a_7-a_8}{a_7+a_8})$, $\epsilon_2 = 0$ and $\epsilon_5 = \frac{a_4}{2\hat{b}a_2} \ln(\frac{a_7-a_8}{a_7+a_8}) + \frac{a_3}{a_2}$ and $\epsilon_6 = \frac{a_4}{a_2}$ to have $\tilde{a_3} = \tilde{a_4} = \tilde{a_8} = 0$ and so $\langle X_2 + \alpha X_5 + \beta X_7 \rangle$, where $\alpha \in \mathbb{R}$, $\beta \neq 0$.*

Case5: *$a_1 = 0$, $a_2 \neq 0$, $a_6 = 0$, $a_7 + a_8 \neq 0$, $\frac{a_7-a_8}{a_7+a_8} < 0$: Let $\epsilon_1 = \frac{1}{2\hat{b}} \ln(-\frac{a_7-a_8}{a_7+a_8})$, $\epsilon_2 = 0$ and $\epsilon_5 = \frac{a_4}{2\hat{b}a_2} \ln(-\frac{a_7-a_8}{a_7+a_8}) + \frac{a_3}{a_2}$, and $\epsilon_6 = \frac{a_4}{a_2}$ to have $\tilde{a_3} = \tilde{a_4} = \tilde{a_7} = 0$ and so we obtain the conjugacy class $\langle X_2 + \alpha X_5 + \beta X_8 \rangle$, where $\alpha \in \mathbb{R}$, $\beta \neq 0$.*

Case6: *$a_1 = 0$, $a_2 \neq 0$, $a_6 = 0$, $a_7 + a_8 \neq 0$, $\frac{a_7-a_8}{a_7+a_8} = 0$: Note that in this case $a_8 \neq 0$, so let $\epsilon_1 = \frac{\ln|a_8|}{\hat{a}-\hat{b}}$, $\epsilon_2 = 0$, $\epsilon_5 = \frac{a_4}{(\hat{a}-\hat{b})a_2} \ln|a_8| + \frac{a_3}{a_2}$ and $\epsilon_6 = \frac{a_4}{a_2}$ to have $\tilde{a_3} = \tilde{a_4} = 0$ and so $\langle \alpha X_2 + \beta X_5 + X_7 + X_8 \rangle$, where $\alpha \neq 0$, $\beta \in \mathbb{R}$.*

Case7: *$a_1 = 0$, $a_2 \neq 0$, $a_6 = 0$, $a_7 + a_8 = 0$, $a_8 \neq 0$: Let $\epsilon_1 = \frac{\ln|a_8|}{\hat{a}+\hat{b}}$, $\epsilon_2 = 0$, $\epsilon_5 = \frac{a_4}{(\hat{a}+\hat{b})a_2} \ln|a_8| + \frac{a_3}{a_2}$ and $\epsilon_6 = \frac{a_4}{a_2}$ to have $\tilde{a_3} = \tilde{a_4} = 0$ and so we obtain the congugacy class $\langle \alpha X_2 + \beta X_5 + X_7 - X_8 \rangle$, where $\alpha \neq 0$, $\beta \in \mathbb{R}$.*

Case8: *$a_1 = 0$, $a_2 \neq 0$, $a_6 = 0$, $a_7 + a_8 = 0$, $a_8 = 0$: Let $\epsilon_1 = \epsilon_2 = 0$, $\epsilon_5 = \frac{a_3}{a_2}$ and $\epsilon_6 = \frac{a_4}{a_2}$ to have $\tilde{a_3} = \tilde{a_4} = 0$ and so the conjugacy class is of the form $\langle X_2 + \alpha X_5 \rangle$, where $\alpha \in \mathbb{R}$.*

Case9 *$a_1 = 0$, $a_2 = 0$, $a_6 \neq 0$: Let $\epsilon_1 = -\frac{a_5}{a_6}$, $\epsilon_2 = -\frac{a_4}{a_6}$ and $\epsilon_6 = 0$ to have $\tilde{a_4} = \tilde{a_5} = 0$ and so the conjugacy class is of the form $\langle \alpha X_3 + X_6 + \beta X_7 + \gamma X_8 \rangle$, where $\alpha, \beta, \gamma \in \mathbb{R}$.*

Case10: *$a_1 = 0$, $a_2 = 0$, $a_6 = 0$, $a_5 \neq 0$, $a_7 + a_8 \neq 0$, $\frac{a_7-a_8}{a_7+a_8} > 0$: Let $\epsilon_1 = \frac{1}{2\hat{b}} \ln(\frac{a_7-a_8}{a_7+a_8})$ and $\epsilon_2 = \frac{a_4}{2\hat{b}a_5} \ln(\frac{a_7-a_8}{a_7+a_8}) + \frac{a_3}{a_5}$ to have $\tilde{a_3} = \tilde{a_8} = 0$ and so the conjugacy class is of the form $\langle \alpha X_4 + X_5 + \beta X_7 \rangle$, where $\alpha \in \mathbb{R}$, $\beta \neq 0$.*

Case11: *$a_1 = 0$, $a_2 = 0$, $a_6 = 0$, $a_5 \neq 0$, $a_7 + a_8 \neq 0$, $\frac{a_7-a_8}{a_7+a_8} < 0$: Let*

$\epsilon_1 = \frac{1}{2\hat{b}} \ln(-\frac{a_7-a_8}{a_7+a_8})$ *and* $\epsilon_2 = -\frac{a_4}{2\hat{b}a_5} \ln(-\frac{a_7-a_8}{a_7+a_8}) - \frac{a_3}{a_5}$ *to have* $\tilde{a_3} = \tilde{a_7} = 0$ *and so the conjugacy class is of the form* $\langle \alpha X_4 + X_5 + \beta X_8 \rangle$*, where* $\alpha \in \mathbb{R}$*,* $\beta \neq 0$*.*

Case12: $a_1 = 0,\ a_2 = 0,\ a_6 = 0,\ a_5 \neq 0,\ a_7 + a_8 \neq 0,\ \frac{a_7-a_8}{a_7+a_8} = 0$ *: Note that in this case* $a_8 \neq 0$*, so let* $\epsilon_1 = \frac{\ln|a_8|}{\hat{a}-\hat{b}}$ *and* $\epsilon_2 = -\frac{a_4}{(\hat{a}-\hat{b})a_5} \ln|a_8| - \frac{a_3}{a_5}$*, to have* $\tilde{a_3} = 0$*, and so the conjugacy class is* $\langle \alpha X_4 + \beta X_5 + X_7 + X_8 \rangle$*, where* $\alpha \in \mathbb{R}$*,* $\beta \neq 0$*.*

Case13: $a_1 = 0,\ a_2 = 0,\ a_6 = 0,\ a_5 \neq 0,\ a_7 + a_8 = 0,\ a_8 \neq 0$ *: Let* $\epsilon_1 = \frac{\ln|a_8|}{\hat{a}+\hat{b}}$ *and* $\epsilon_2 = \frac{a_4}{(\hat{a}+\hat{b})a_5} \ln|a_8| + \frac{a_3}{a_5}$ *to have* $\tilde{a_3} = 0$ *and so the conjugacy class is of the form* $\langle \alpha X_4 + \beta X_5 + X_7 - X_8 \rangle$*, where* $\alpha \in \mathbb{R}$*,* $\beta \neq 0$*.*

Case14: $a_1 = 0,\ a_2 = 0,\ a_6 = 0,\ a_5 \neq 0,\ a_7 + a_8 = 0,\ a_8 = 0$ *: Let* $\epsilon_1 = 0$ *and* $\epsilon_2 = -\frac{a_3}{a_5}$ *to have* $\tilde{a_3} = 0$ *and so we obtain the conjugacy class* $\langle \alpha X_4 + X_5 \rangle$*,* $\alpha \in \mathbb{R}$*.*

Case15: $a_1 = 0,\ a_2 = 0,\ a_6 = 0,\ a_5 = 0,\ a_4 \neq 0$ *: let* $\epsilon_1 = -\frac{a_3}{a_4}$ *to have* $\tilde{a_3} = 0$ *and so the conjugacy class is* $\langle X_4 + \alpha X_7 + \beta X_8 \rangle$*, where* $\alpha, \beta \in \mathbb{R}$*.*

Case16: $a_1 = 0,\ a_2 = 0,\ a_6 = 0,\ a_5 = 0,\ a_4 = 0,\ a_7 + a_8 \neq 0,\ \frac{a_7-a_8}{a_7+a_8} > 0$ *: Let* $\epsilon_1 = \frac{1}{2\hat{b}} \ln(\frac{a_7-a_8}{a_7+a_8})$ *to have* $\tilde{a_8} = 0$ *and so the conjugacy class after appropriate scaling is of the form* $\langle \alpha X_3 + X_7 \rangle$*, where* $\alpha \in \mathbb{R}$*.*

Case17: $a_1 = 0,\ a_2 = 0,\ a_6 = 0,\ a_5 = 0,\ a_4 = 0,\ a_7 + a_8 \neq 0,\ \frac{a_7-a_8}{a_7+a_8} < 0$ *: Let* $\epsilon_1 = \frac{1}{2\hat{b}} \ln(-\frac{a_7-a_8}{a_7+a_8})$ *to have* $\tilde{a_7} = 0$ *and so the conjugacy class after appropriate scaling is of the form* $\langle \alpha X_3 + X_8 \rangle$*, where* $\alpha \in \mathbb{R}$*.*

Case18: $a_1 = 0,\ a_2 = 0,\ a_6 = 0,\ a_5 = 0,\ a_4 = 0,\ a_7 + a_8 \neq 0,\ \frac{a_7-a_8}{a_7+a_8} = 0$ *: Note that* $a_8 \neq 0$ *so let* $\epsilon_1 = \frac{\ln|a_8|}{\hat{a}-\hat{b}}$ *to have thr conjugacy class* $\langle \alpha X_3 + X_7 + X_8 \rangle$*, where* $\alpha \in \mathbb{R}$*.*

Case19: $a_1 = 0,\ a_2 = 0,\ a_6 = 0,\ a_5 = 0,\ a_4 = 0,\ a_7 + a_8 = 0,\ a_8 \neq 0$ *: Let*

$\epsilon_1 = \frac{\ln|a_8|}{\hat{a}+\hat{b}}$ *to have the conjugacy class* $\langle \alpha X_3 + X_7 - X_8 \rangle$, *where* $\alpha \in \mathbb{R}$.

Case20: $a_1 = 0,\ a_2 = 0,\ a_6 = 0,\ a_5 = 0,\ a_4 = 0,\ a_7 + a_8 = 0,\ a_8 = 0$ *: Then directly we have the conjugacy class* $\langle X_3 \rangle$.

∎

4.3.3 Optimal system for the case $d^2 - 4k\rho_2 = -\mu^2$

The non-zero commutators of the Lie algebra $\mathcal{L}^8$ with basis (4.2.22) are given by

$$
\begin{array}{lll}
[X_1, X_4] = X_3, & [X_1, X_6] = X_5\ , & [X_1, X_7] = -\frac{d}{2\rho_2}X_7 - \frac{\mu}{2\rho_2}X_8, \\
[X_1, X_8] = -\frac{d}{2\rho_2}X_8 + \frac{\mu}{2\rho_2}X_7, & [X_2, X_5] = X_3, & [X_2, X_6] = X_4.
\end{array}
\tag{4.3.9}
$$

The Lie algebra $\mathcal{L}^8$ is solvable and the Killing form is given by $K = 2(\hat{a}^2 - \hat{b}^2)a_1^2$ where $\hat{a} = \frac{d}{2\rho_2}$ and $\hat{b} = \frac{\mu}{2\rho_2}$. The adjoint table is given by

Table 4.3: The adjoint table corresponding to commutators (4.3.9)

$Ad(e^{\epsilon})$	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
X_1	X_1	X_2	X_3	$X_4 - \epsilon X_3$	X_5	$X_6 - \epsilon X_5$	Y_1	Y_2
X_2	X_1	X_2	X_3	X_4	$X_5 - \epsilon X_3$	$X_6 - \epsilon X_4$	X_7	X_8
X_3	X_2	X_2	X_3	X_4	X_5	X_6	X_7	X_8
X_4	$X_1 + \epsilon X_3$	X_2	X_3	X_4	X_5	X_6	X_7	X_8
X_5	X_1	$X_2 + \epsilon X_3$	X_3	X_4	X_5	X_6	X_7	X_8
X_6	$X_1 + \epsilon X_5$	$X_2 + \epsilon X_4$	X_3	X_4	X_5	X_6	X_7	X_8
X_7	$X_1 - \epsilon\hat{a}X_7 - \epsilon\hat{b}X_8$	X_2	X_3	X_4	X_5	X_6	X_7	X_8
X_8	$X_1 + \epsilon\hat{b}X_7 - \epsilon\hat{a}X_8$	X_2	X_3	X_4	X_5	X_6	X_7	X_8

$Y_1 = e^{\epsilon\hat{a}}(\cos(\epsilon\hat{b})X_7 + \sin(\epsilon\hat{b})X_8)$ and $Y_2 = e^{\epsilon\hat{a}}(-\sin(\epsilon\hat{b})X_7 + \cos(\epsilon\hat{b})X_8)$.

The adjoint group is defined by the matrix

$$A = Ad(e^{-\epsilon_8 X_8}).Ad(e^{-\epsilon_7 X_7}). \ ... \ .Ad(e^{-\epsilon_1 X_1}),$$

which is given by

$$A = \begin{pmatrix} 1 & 0 & -\epsilon_4 & 0 & -\epsilon_6 & 0 & \hat{a}\epsilon_7 - \hat{b}\epsilon_8 & \hat{a}\epsilon_8 + \hat{b}\epsilon_7 \\ 0 & 1 & -\epsilon_5 & -\epsilon_6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \epsilon_1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \epsilon_2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \epsilon_1\epsilon_2 & \epsilon_2 & \epsilon_1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \hat{Y}_1 & -\hat{Y}_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & \hat{Y}_2 & \hat{Y}_1 \end{pmatrix}. \quad (4.3.10)$$

$$\hat{Y}_1 = e^{-\epsilon_1 \hat{a}} \cos(\epsilon_1 \hat{b}), \quad \hat{Y}_2 = e^{-\epsilon_1 \hat{a}} \sin(\epsilon_1 \hat{b})$$

Theorem 4.3.3 *An optimal system of one-dimensional Lie algebra L^8 with basis (4.2.22) is provided by the following generators*

$$\begin{array}{ll|ll} X^1 = X_1 + \alpha X_2 + \beta X_6, & \alpha \in \mathbb{R}, \beta \neq 0, & X^6 = X_4 + \alpha X_5 + \beta X_7 + \gamma X_8, & \alpha, \beta, \gamma \in \mathbb{R}, \\ X^2 = X_1 + \alpha X_2 + \beta X_4, & \alpha, \beta \in \mathbb{R}, & X^7 = X_5 + \alpha X_8, & \alpha \neq 0, \\ X^3 = X_2 + \alpha X_6 + \beta X_7 + \gamma X_8, & \alpha \neq 0, \beta, \gamma \in \mathbb{R}, & X^8 = X_5 + \alpha X_7, & \alpha \in \mathbb{R}, \\ X^4 = X_2 + \alpha X_5 + \beta X_7 + \gamma X_8, & \alpha, \beta, \gamma \in \mathbb{R}, & X^9 = \alpha X_3 + X_8, & \alpha \in \mathbb{R}, \\ X^5 = \alpha X_3 + X_6 + \beta X_7 + \gamma X_8, & \alpha, \beta, \gamma \in \mathbb{R}, & X^{10} = \alpha X_3 + \beta X_7, & \alpha, \beta \in \mathbb{R}. \end{array} \quad (4.3.11)$$

Proof. *Let X and $\tilde{X}$ be two elements in the Lie algebra $\mathcal{L}^8$ with basis (4.2.22) given by $X = \sum_{i=1}^{8} a_i X_i$ and $\tilde{X} = \sum_{i=1}^{8} \tilde{a}_i X_i$. For simplicity, we will write X and $\tilde{X}$ as row vectors of the coefficients on the form $X = (\, a_1 \; a_2 \; ... \; a_8 \,)$ and $\tilde{X} = (\, \tilde{a_1} \; \tilde{a_2} \; ... \; \tilde{a_8} \,)$. Then in order for X and $\tilde{X}$ to be in the same conjugacy class, we must have $\tilde{X} = XA$, where A is given by (4.3.10). So, the theorem is proved by solving the system*

$$
\begin{aligned}
\tilde{a_1} &= a_1, \\
\tilde{a_2} &= a_2, \\
\tilde{a_3} &= a_3 + \epsilon_1 a_4 + \epsilon_2 a_5 - \epsilon_5 a_2 - \epsilon_4 a_1 + \epsilon_1 \epsilon_2 a_6, \\
\tilde{a_4} &= a_4 - \epsilon_6 a_2 + \epsilon_2 a_6, \\
\tilde{a_5} &= a_5 - \epsilon_6 a_1 + \epsilon_1 a_6, \\
\tilde{a_6} &= a_6, \\
\tilde{a_7} &= a_1(\hat{a}\epsilon_7 - \hat{b}\epsilon_8) + e^{-\epsilon_1 \hat{a}} \left(\cos(\epsilon_1 \hat{b}) a_7 + \sin(\epsilon_1 \hat{b}) a_8 \right), \\
\tilde{a_8} &= a_1(\hat{b}\epsilon_7 + \hat{a}\epsilon_8) - e^{-\epsilon_1 \hat{a}} \left(\sin(\epsilon_1 \hat{b}) a_7 - \cos(\epsilon_1 \hat{b}) a_8 \right),
\end{aligned}
\tag{4.3.12}
$$

for $\{\epsilon_i\}_{i=1}^{8}$ in term of $\{a_i\}_{i=1}^{8}$ in order to get the simplest values of $\{\tilde{a}_i\}_{i=1}^{8}$. The results are presented for different cases in the tree diagram 4.2, where it is initiated by the sign of the Killing form and its leafs are given completely.

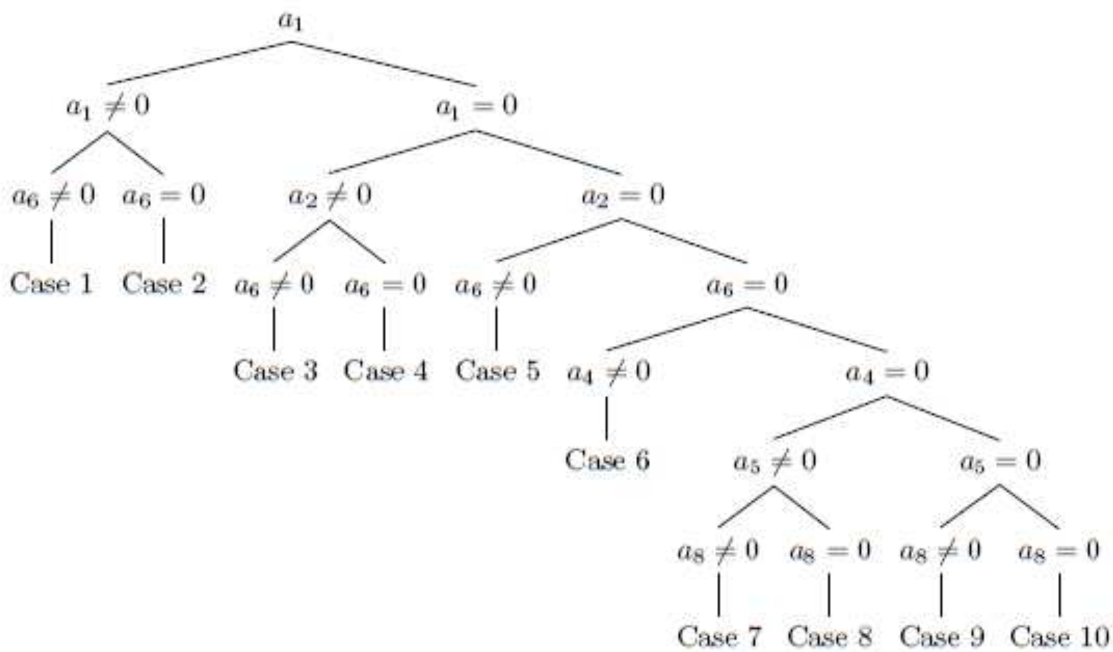


Figure 4.2: Tree diagram

The full details for each leaf are given as follows:

Case1: $a_1 \neq 0,\ a_6 \neq 0$ *: Let* $\epsilon_1 = -\frac{a_5}{a_6}$, $\epsilon_2 = -\frac{a_4}{a_6}$, $\epsilon_4 = \frac{a_3 a_6 - a_4 a_5}{a_1 a_6}$, $\epsilon_5 = \epsilon_6 = 0$, $\epsilon_7 = \frac{1}{a_1(\hat{a}^2+\hat{b}^2)} e^{\frac{a_5 \hat{a}}{a_6}} \left(\left(\hat{a} a_8 - \hat{b} a_7\right) \sin\left(\frac{a_5 \hat{b}}{a_6}\right) - \left(\hat{a} a_7 + \hat{b} a_8\right) \cos\left(\frac{a_5 \hat{b}}{a_6}\right)\right)$ *and* $\epsilon_8 = -\frac{1}{a_1(\hat{a}^2+\hat{b}^2)} e^{\frac{a_5 \hat{a}}{a_6}} \left(\left(\hat{a} a_7 + \hat{b} a_8\right) \sin\left(\frac{a_5 \hat{b}}{a_6}\right) + \left(\hat{a} a_8 - \hat{b} a_7\right) \cos\left(\frac{a_5 \hat{b}}{a_6}\right)\right)$ *to have* $\tilde{a_3} = \tilde{a_4} = \tilde{a_5} = \tilde{a_7} = \tilde{a_8} = 0$ *then we have the conjugacy class* $\langle X_1 + \alpha X_2 + \beta X_6 \rangle$, *with* $\alpha \in \mathbb{R}$, *and* $\beta \neq 0$.

Case2: $a_1 \neq 0,\ a_6 = 0$ *: Let* $\epsilon_1 = \epsilon_2 = 0$, $\epsilon_4 = \frac{a_3}{a_1}$, $\epsilon_6 = \frac{a_5}{a_1}$, $\epsilon_7 = -\frac{\hat{a} a_7 + \hat{b} a_8}{a_1(\hat{a}^2+\hat{b}^2)}$, *and* $\epsilon_8 = -\frac{\hat{a} a_8 - \hat{b} a_7}{a_1(\hat{a}^2+\hat{b}^2)}$ *to have* $\tilde{a_3} = \tilde{a_5} = \tilde{a_7} = \tilde{a_8} = 0$. *Then we have the conjugacy class of the form* $\langle X_1 + \alpha X_2 + \beta X_4 \rangle$ *where* $\alpha, \beta \in \mathbb{R}$.

Case3: $a_1 = 0,\ a_2 \neq 0,\ a_6 \neq 0$ *: Let* $\epsilon_1 = -\frac{a_5}{a_6}$, $\epsilon_2 = 0$, $\epsilon_5 = \frac{a_3 a_6 - a_4 a_5}{a_2 a_6}$, *and* $\epsilon_6 = \frac{a_4}{a_2}$ *to get* $\tilde{a_3} = \tilde{a_4} = \tilde{a_5} = 0$. *Then the conjugacy class is of the form* $\langle X_2 + \alpha X_6 + \beta X_7 + \gamma X_8 \rangle$, *where* $\alpha \neq 0$, $\beta, \gamma \in \mathbb{R}$.

Case4: *$a_1 = 0,\ a_2 \neq 0,\ a_6 = 0$: Let $\epsilon_1 = \epsilon_2 = 0,\ \epsilon_5 = \frac{a_3}{a_2},\ \epsilon_6 = \frac{a_4}{a_2}$ to get $\tilde{a_3} = \tilde{a_4} = 0$. Then the conjugacy class is $\langle X_2 + \alpha X_5 + \beta X_7 + \gamma X_8\rangle,\ \alpha, \beta, \gamma \in \mathbb{R}$.*

Case5: *$a_1 = 0,\ a_2 = 0,\ a_6 \neq 0$: Let $\epsilon_1 = -\frac{a_5}{a_6},\ \epsilon_2 = -\frac{a_4}{a_6}$ to have $\tilde{a_4} = \tilde{a_5} = 0$ and so we obtain the conjugacy class $\langle \alpha X_3 + X_6 + \beta X_7 + \gamma X_8\rangle,\ \alpha, \beta, \gamma \in \mathbb{R}$.*

Case6: *$a_1 = 0,\ a_2 = 0,\ a_6 = 0,\ a_4 \neq 0$: Let $\epsilon_1 = -\frac{a_3}{a_4}$ to have $\tilde{a_3} = 0$ and so we obtain the conjugacy class $\langle X_4 + \alpha X_5 + \beta X_7 + \gamma X_8\rangle,\ \alpha, \beta, \gamma \in \mathbb{R}$.*

Case7: *$a_1 = 0,\ a_2 = 0,\ a_6 = 0,\ a_4 = 0\ a_5 \neq 0,\ a_8 \neq 0$: Let $\epsilon_1 = -\frac{1}{b}\tan^{-1}(\frac{a_7}{a_8})$ and $\epsilon_2 = -\frac{a_3}{a_5}$ to have $\tilde{a_3} = \tilde{a_7} = 0$. Then the conjugacy class is of the form $\langle X_5 + \alpha X_8\rangle$, where $\alpha \neq 0$.*

Case8: *$a_1 = 0,\ a_2 = 0,\ a_6 = 0,\ a_4 = 0\ a_5 \neq 0,\ a_8 = 0$: Let $\epsilon_1 = 0$ and $\epsilon_2 = -\frac{a_3}{a_5}$ to have $\tilde{a_3} = \tilde{a_8} = 0$. Then the conjugacy class is $\langle X_5 + \alpha X_7\rangle,\ \alpha \in \mathbb{R}$.*

Case9: *$a_1 = 0,\ a_2 = 0,\ a_6 = 0,\ a_4 = 0,\ a_5 = 0,\ a_8 \neq 0$: Let $\epsilon_1 = -\frac{1}{b}\tan^{-1}(\frac{a_7}{a_8})$ to have $\tilde{a_7} = 0$. By appropriate scaling the conjugacy class is of the form $\langle \alpha X_3 + X_8\rangle$, where $\alpha \in \mathbb{R}$.*

Case10: *$a_1 = 0,\ a_2 = 0,\ a_6 = 0,\ a_4 = 0,\ a_5 = 0,\ a_8 = 0$: Let $\epsilon_1 = 0$ to have $\tilde{a_8} = 0$ and so we have the conjugacy class $\langle \alpha X_3 + \beta X_7\rangle$, where $\alpha, \beta \in \mathbb{R}$.*

∎

4.4 Optimal reductions and invariant solutions

It is known that the invariant solutions for PDEs can be determined by two procedures, which are the invariant form method and the direct substitution method [5]. The idea of looking for group invariant solutions generalize quite naturally to

PDEs with any number of independent and dependent variables. A one parameter group that acts nontrivially on one or more independent variables can be used to reduce the number of independent variables by one.

In this section, we focus on the invariant form method which requires that at least one of the infinitesimals ξ^1 and ξ^2 does not equal zero [5, 33]. Hence, we solve the invariance surface conditions explicitly by solving the corresponding characteristic equation given by

$$\frac{dt}{\xi^1(t,x,\varphi,\psi)} = \frac{dx}{\xi^2(t,x,\varphi,\psi)} = \frac{d\varphi}{\eta^1(t,x,\varphi,\psi)} = \frac{d\psi}{\eta^2(t,x,\varphi,\psi)}, \tag{4.4.1}$$

to get the corresponding invariants which are used to reduce the number of independent variables by one. The procedure is explained in details in three examples. Moreover, all possible invariant variables of the optimal systems (4.3.3), (4.3.7) and (4.3.11) and their corresponding reductions for the three non-linear cases of $\chi(\psi_x)$ are given in Table 4.4, Table 4.5 and Table 4.6, respectively.

Example 4.4.1 *Reduction for Case1: $d^2 - 4k\rho_2 = 0$, using invariant form of X^3. Consider the generator $X^3 = X_2 + \alpha X_6 + \beta X_7 + \gamma X_8$ where $\alpha \neq 0, \beta, \gamma \in \mathbb{R}$, from the optimal system (4.3.3). Solving the corresponding characteristic equation*

$$\frac{dt}{0} = \frac{dx}{1} = \frac{d\varphi}{\alpha t x} = \frac{d\psi}{\alpha(\frac{d}{k} - t) + e^{-\frac{d}{2\rho_2}t}(\beta + \gamma t)},$$

gives the invariant variables as follows:

$$
\begin{aligned}
\phi(t,x) \;&=\; Z(\zeta) + \tfrac{\alpha}{2} t x^2, & \zeta \;&=\; t, \\
\psi(t,x) \;&=\; W(\zeta) + \alpha x(\tfrac{d}{k} - t) + x(\beta + \gamma t) e^{-\frac{d}{2\rho_2} t}.
\end{aligned}
\tag{4.4.2}
$$

The reduced system resulting from the invariant variables (4.4.2) is the system of ODEs of the form

$$
\begin{aligned}
\rho_1 \, Z'' - k(\gamma\zeta + \beta) e^{-\frac{d}{2\rho_2}\zeta} - \alpha d \;&=\; 0, \\
\rho_2 \, W'' + d\, W' + k\, W \;&=\; 0.
\end{aligned}
$$

Solving this system yields the following solution

$$
\begin{aligned}
\varphi(t,x) &= \tfrac{\rho_2}{\rho_1 d}\left(d\gamma t + d\beta + 4\gamma\rho_2\right) e^{-\frac{d}{2\rho_2} t} + \alpha t x^2 + \tfrac{\alpha d}{\rho_1} t^2 + 2c_1 t + 2c_2, \\
\psi(t,x) &= (\gamma t x + \beta x + c_3 t + c_4) e^{-\frac{d}{2\rho_2} t} - \alpha t x + \tfrac{\alpha d}{k} x,
\end{aligned}
$$

where $d = 2\sqrt{k\rho_2}$.

Example 4.4.2 *Reduction for Case2:* $d^2 - 4k\rho_2 = \lambda^2$ *using the invariant* X^3. *Consider the generator* $X^3 = X_2 + \alpha X_6 + \beta X_7 + \gamma X_8$ *where* $\alpha \neq 0, \beta, \gamma \in \mathbb{R}$, *from the optimal system (4.3.7). Solving the corresponding characteristic equation*

$$
\frac{dt}{0} = \frac{dx}{1} = \frac{d\varphi}{\alpha t x} = \frac{d\psi}{\alpha(\frac{d}{k} - t) + e^{-\frac{d}{2\rho_2} t}\left(\beta \cosh\left(\frac{\lambda\, t}{2\rho_2}\right) + \gamma \sinh\left(\frac{\lambda\, t}{2\rho_2}\right)\right)},
$$

gives the invariant variables as follows:

$$
\begin{aligned}
\phi(t,x) \;&=\; Z(\zeta) + \tfrac{\alpha}{2} t x^2, \qquad\qquad\qquad \zeta \;=\; t,\\
\psi(t,x) \;&=\; W(\zeta) + \alpha x(\tfrac{d}{k} - t) + x e^{-\frac{d}{2\rho_2} t}(\beta \cosh(\tfrac{\lambda\, t}{2\rho_2}) + \gamma \sinh(\tfrac{\lambda\, t}{2\rho_2})).
\end{aligned} \tag{4.4.3}
$$

The reduced system resulting from the invariant variables (4.4.3) is the system of ODEs of the form

$$
\begin{aligned}
&\rho_1\, Z'' - k\, e^{-\frac{d}{2\rho_2}\zeta}(\beta \cosh(\tfrac{\lambda\zeta}{2\rho_2}) + \gamma \sinh(\tfrac{\lambda\zeta}{2\rho_2})) - \alpha d \;=\; 0,\\
&\rho_2\, W'' + d\, W' + k\, W \;=\; 0.
\end{aligned}
$$

Solving this system yields the following solution

$$
\begin{aligned}
\varphi(t,x) \;&=\; \tfrac{1}{2k\rho_1}\left((2\,\rho_2 \beta\, k + \beta\,\lambda^2 + \lambda d \gamma)\cosh\left(\tfrac{\lambda\, t}{2\rho_2}\right)\right) \mathrm{e}^{-\frac{d}{2\rho_2} t}\\
&+\; \tfrac{1}{2k\rho_1}\left((2\,\gamma\,\rho_2 k + \,\lambda\beta\, d + \gamma\,\lambda^2)\sinh\left(\tfrac{\lambda\, t}{2\rho_2}\right)\right) \mathrm{e}^{-\frac{d}{2\rho_2} t}\\
&+\; \tfrac{\alpha\, d}{2\rho_1} t^2 + \tfrac{\alpha}{2}\, t\, x^2 +\, c_1 t + c_2,\\
\psi(t,x) \;&=\; \left((c_3 + \beta x)\cosh\left(\tfrac{\lambda\, t}{2\rho_2}\right) + (c_4 + \gamma x)\sinh\left(\tfrac{\lambda\, t}{2\rho_2}\right)\right) \mathrm{e}^{-\frac{d}{2\rho_2} t} - x\alpha\, t + \tfrac{\alpha\, d}{k} x,
\end{aligned}
$$

where $d = \sqrt{k\rho_2 + \lambda^2}$.

Example 4.4.3 *Reduction for Case3:* $d^2 - 4k\rho_2 = -\mu^2$ *using the invariant* X^3. *Consider the generator* $X^3 = X_2 + \alpha X_6 + \beta X_7 + \gamma X_8$ *where* $\alpha \neq 0, \beta, \gamma \in \mathbb{R}$, *from the optimal system (4.3.11). Solving the corresponding characteristic equation*

$$
\frac{dt}{0} = \frac{dx}{1} = \frac{d\varphi}{\alpha t x} = \frac{d\psi}{\alpha(\frac{d}{k} - t) + e^{-\frac{d}{2\rho_2} t}\left(\beta \cos\left(\frac{\mu t}{2\rho_2}\right) + \gamma \sin\left(\frac{\mu t}{2\rho_2}\right)\right)},
$$

gives the invariant variables as follows:

$$\begin{aligned}
\phi(t,x) &= Z(\zeta) + \tfrac{\alpha}{2} t x^2, \qquad\qquad \zeta \;=\; t, \\
\psi(t,x) &= W(\zeta) + \alpha x(\tfrac{d}{k} - t) + x e^{-\frac{d}{2\rho_2}t}\left(\beta\cos(\tfrac{\mu t}{2\rho_2}) + \gamma\sin(\tfrac{\mu t}{2\rho_2})\right).
\end{aligned} \tag{4.4.4}$$

The reduced system resulting from the invariant variables (4.4.4) is the system of ODEs of the form

$$\begin{aligned}
&\rho_1 Z'' - k e^{-\frac{d}{2\rho_2}\zeta}\left(\beta\cos(\tfrac{\mu\zeta}{2\rho_2}) + \gamma\sin(\tfrac{\mu\zeta}{2\rho_2})\right) - \alpha d = 0, \\
&\rho_2 W'' + d\,W' + k\,W = 0.
\end{aligned}$$

Solving this system yields the following solution

$$\begin{aligned}
\varphi(t,x) = \;& \left((\beta\,\mu^2 - 2\,\beta\,k\rho_2 - d\gamma\,\mu)\cos\left(\frac{\mu\,t}{2\rho_2}\right) + (\gamma\,\mu^2 - 2\,\gamma\,k\rho_2)\sin\left(\frac{\mu\,t}{2\rho_2}\right)\right)\mathrm{e}^{-\frac{dt}{2\rho_2}} \\
& -\frac{\beta\mu d}{2\rho_2}\;(-\alpha\,kx^2\rho_1\rho_2 + d)\,t\sin\left(\frac{\mu\,t}{2\rho_2}\right) - (\alpha\,kdt^2 + 2\,c_1\,\rho_1 kt + 2\,c_2)\,, \\
\psi(t,x) = \;& \left((\beta x + c_3)\cos\left(\frac{\mu\,t}{2\rho_2}\right) + (\gamma x + c_4)\sin\left(\frac{\mu\,t}{2\rho_2}\right)\right)\mathrm{e}^{-\frac{dt}{2\rho_2}} + \alpha\,x\left(\tfrac{d}{k} - t\right),
\end{aligned} \tag{4.4.5}$$

where $d = \sqrt{k\rho_2 - \mu^2}$.

Table 4.4: Reduction using one-dimensional optimal system (4.3.3)

Generators in (??)	Invariant variables	The reduced system
$X^1 = X_1 + \alpha X_2 + \beta X_6$, $\alpha \in \mathbb{R}, \beta \neq 0.$	A	$(k-\alpha^2\rho_1)\,Z'' + k\,W' - \beta\rho_1\,\zeta \;=\; 0,$ $(\chi'(W') - \alpha^2\rho_2)\,W'' - k\,Z' + \alpha d\,W' - k\,W - 3\beta\rho_2 \;=\; 0.$
$X^2 = X_1 + \alpha X_2 + \beta X_4$ $\alpha, \beta \in \mathbb{R}.$	B	$(k-\alpha^2\rho_1)\,Z'' + k\,W' - \beta\rho_1 \;=\; 0,$ $(\chi'(W') - \alpha^2\rho_2)\,W'' - kZ' + \alpha d\,W' - k\,W = 0.$
$X^3 = X_2 + \alpha X_6 + \beta X_7 + \gamma X_8$ $\alpha \neq 0, \beta, \gamma \in \mathbb{R}.$	C	$\rho_1\,Z'' - k(\gamma\zeta + \beta)e^{-\frac{d}{2\rho_2}\zeta} - \alpha d \;=\; 0,$ $\rho_2\,W'' + d\,W' + k\,W \;=\; 0.$
$X^4 = X_2 + \alpha X_5 + \beta X_8$ $\alpha \in \mathbb{R}, \beta \neq 0.$	D	$\rho_1\,Z'' - \beta k\,\zeta\,e^{-\frac{d}{2\rho_2}\zeta} \;=\; 0,$ $\rho_2\,W'' + d\,W' + k\,W \;=\; 0.$
$X^5 = \alpha X_2 + \beta X_5 + X_7$ $\alpha \neq 0, \beta \in \mathbb{R}.$	E	$\rho_1\,Z'' - \frac{k}{\alpha}\,e^{-\frac{d}{2\rho_2}\zeta} \;=\; 0,$ $\rho_2\,W'' + d\,W' + k\,W \;=\; 0.$
$X^6 = X_2 + \alpha X_5$ $\alpha \in \mathbb{R}.$	F	$Z'' \;=\; 0,$ $\rho_2\,W'' + d\,W' + k\,W \;=\; 0.$

$A:$	$\phi(t,x) \;=\; Z(\zeta) + \frac{\beta}{2}t^2(x - \frac{\alpha}{3}t),$	$\psi(t,x) \;=\; W(\zeta) - \frac{\beta}{2}t^2 + \frac{\beta d}{k}t,$	$\zeta \;=\; x - \alpha t.$
$B:$	$\phi(t,x) \;=\; Z(\zeta) + \frac{\beta}{2}t^2,$	$\psi(t,x) \;=\; W(\zeta),$	$\zeta \;=\; x - \alpha t.$
$C:$	$\phi(t,x) \;=\; Z(\zeta) + \frac{\alpha}{2}tx^2,$	$\psi(t,x) \;=\; W(\zeta) + \alpha x(\frac{d}{k} - t) + x(\beta + \gamma t)e^{-\frac{d}{2\rho_2}t},$	$\zeta \;=\; t.$
$D:$	$\phi(t,x) \;=\; Z(\zeta) + \frac{\alpha}{2}x^2,$	$\psi(t,x) = W(\zeta) - x(\alpha - \beta t\,e^{-\frac{d}{2\rho_2}t}),$	$\zeta \;=\; t.$
$E:$	$\phi(t,x) \;=\; Z(\zeta) + \frac{\beta}{2\alpha}x^2,$	$\psi(t,x) \;=\; W(\zeta) - \frac{x}{\alpha}(\beta - \,e^{-\frac{d}{2\rho_2}t}),$	$\zeta \;=\; t.$
$F:$	$\phi(t,x) \;=\; Z(\zeta) + \frac{\alpha}{2}x^2,$	$\psi(t,x) \;=\; W(\zeta) - \alpha x,$	$\zeta \;=\; t.$

Table 4.5: Reduction using one-dimensional optimal system (4.3.7) with

Generators in (??)	Invariant variables	The reduced system
$X^1 = X_1 + \alpha X_2 + \beta X_6$, $\alpha \in \mathbb{R}, \beta \neq 0.$	A	$(k - \alpha^2\rho_1)\, Z'' + k\, W' - \beta\rho_1\, \zeta \;=\; 0,$ $(\chi'(W') - \alpha^2\rho_2)\, W'' - k\, Z' + \alpha d\, W' - k\, W - \beta(3\rho_2 - \frac{\lambda^2}{k}) \;=\; 0.$
$X^2 = X_1 + \alpha X_2 + \beta X_4$, $\alpha, \beta \in \mathbb{R}.$	B	$(k - \alpha^2\rho_1)\, Z'' + k\, W' - \beta\rho_1 \;=\; 0,$ $(\chi'(W') - \alpha^2\rho_2)\, W'' - k\, Z' + \alpha d\, W' - k\, W = 0.$
$X^3 = X_2 + \alpha X_6 + \beta X_7 + \gamma X_8$ $\alpha \neq 0, \beta, \gamma \in \mathbb{R}.$	C	$\rho_1\, Z'' - k\, e^{-\frac{d}{2\rho_2}\zeta}(\beta\cosh(\frac{\lambda\zeta}{2\rho_2}) + \gamma\sinh(\frac{\lambda\zeta}{2\rho_2})) - \alpha d \;=\; 0,$ $\rho_2\, W'' + d\, W' + k\, W \;=\; 0.$
$X^4 = X_2 + \alpha X_5 + \beta X_7$ $\alpha \in \mathbb{R}, \beta \neq 0.$	D	$\rho_1 Z'' - \beta k\, e^{-\frac{d}{2\rho_2}\zeta}\, \cosh(\frac{\lambda\zeta}{2\rho_2}) \;=\; 0,$ $\rho_2\, W'' + d\, W' + k\, W \;=\; 0.$
$X^5 = X_2 + \alpha X_5 + \beta X_8$ $\alpha \in \mathbb{R}, \beta \neq 0.$	E	$\rho_1\, Z'' - \beta k\, e^{-\frac{d}{2\rho_2}\zeta}\, \sinh(\frac{\lambda\zeta}{2\rho_2}) \;=\; 0,$ $\rho_2\, W'' + d\, W' + k\, W \;=\; 0.$
$X^6 = \alpha X_2 + \beta X_5 + X_7 + X_8$ $\alpha \neq 0, \beta \in \mathbb{R}.$	F	$\rho_1\, Z'' - \frac{k}{\alpha}\, e^{-\frac{d}{2\rho_2}\zeta}(\cosh(\frac{\lambda\zeta}{2\rho_2}) + \sinh(\frac{\lambda\zeta}{2\rho_2})) \;=\; 0,$ $\rho_2\, W'' + d\, W' + k\, W \;=\; 0.$
$X^7 = \alpha X_2 + \beta X_5 + X_7 - X_8$ $\alpha \neq 0, \beta \in \mathbb{R}.$	G	$\rho_1\, Z'' - \frac{k}{\alpha} e^{-\frac{d}{2\rho_2}\zeta}(\cosh(\frac{\lambda\zeta}{2\rho_2}) - \sinh(\frac{\lambda\zeta}{2\rho_2})) \;=\; 0,$ $\rho_2\, W'' + d\, W' + k\, W \;=\; 0.$
$X^8 = X_2 + \alpha X_5$ $\alpha \in \mathbb{R}.$	H	$Z'' \;=\; 0,$ $\rho_2\, W'' + d\, W' + k\, W \;=\; 0.$

$A: \quad \phi(t,x) \;=\; Z(\zeta) - \frac{\beta}{6}t^2(\alpha t - 3x),$	$\psi(t,x) \;=\; W(\zeta) - \frac{\beta}{2}t(t - \frac{2d}{k}),$	$\zeta \;=\; x - \alpha t.$
$B: \quad \phi(t,x) \;=\; Z(\zeta) + \frac{\beta}{2}t^2,$	$\psi(t,x) \;=\; W(\zeta),$	$\zeta \;=\; x - \alpha t.$
$C: \quad \phi(t,x) \;=\; Z(\zeta) + \frac{\alpha}{2}tx^2,$	$\psi(t,x) \;=\; W(\zeta) + \alpha x(\frac{d}{k} - t) + xe^{-\frac{d}{2\rho_2}t}(\beta\cosh(\frac{\lambda t}{2\rho_2}) + \gamma\sinh(\frac{\lambda t}{2\rho_2})),$	$\zeta \;=\; t.$
$D: \quad \phi(t,x) \;=\; Z(\zeta) + \frac{\alpha}{2}x^2,$	$\psi(t,x) \;=\; W(\zeta) - x(\,\alpha - \beta e^{-\frac{d}{2\rho_2}t}\cosh(\frac{\lambda t}{2\rho_2})\,),$	$\zeta \;=\; t.$
$E: \quad \phi(t,x) \;=\; Z(\zeta) + \frac{\alpha}{2}x^2,$	$\psi(t,x) \;=\; W(\zeta) - x(\,\alpha - \beta e^{-\frac{d}{2\rho_2}t}\sinh(\frac{\lambda t}{2\rho_2})\,),$	$\zeta \;=\; t.$
$F: \quad \phi(t,x) \;=\; Z(\zeta) + \frac{\beta}{2\alpha}x^2,$	$\psi(t,x) \;=\; W(\zeta) + \frac{x}{\alpha}e^{-\frac{d}{2\rho_2}t}(\cosh(\frac{\lambda t}{2\rho_2}) + \sinh(\frac{\lambda t}{2\rho_2}) - \beta),$	$\zeta \;=\; t.$
$G: \quad \phi(t,x) \;=\; Z(\zeta) + \frac{\beta}{2\alpha}x^2,$	$\psi(t,x) \;=\; W(\zeta) + \frac{x}{\alpha}e^{-\frac{d}{2\rho_2}t}(\cosh(\frac{\lambda t}{2\rho_2}) - \sinh(\frac{\lambda t}{2\rho_2}) - \beta),$	$\zeta \;=\; t.$
$H: \quad \phi(t,x) \;=\; Z(\zeta) + \frac{\alpha}{2}x^2,$	$\psi(t,x) = W(\zeta) - \alpha x,$	$\zeta \;=\; t.$

Table 4.6: Reduction using one-dimensional optimal system (4.3.11)

Generators in (??)	Invariant variables	The reduced system
$X^1 = X_1 + \alpha X_2 + \beta X_6$ $\alpha \in \mathbb{R}, \beta \neq 0.$	A	$(k - \alpha^2\rho_1)\, Z'' + k\, W' - \beta\rho_1\, \zeta \;=\; 0,$ $(\chi'(W') - \alpha^2\rho_2)\, W'' - k\, Z' + \alpha d\, W' - k\, W + \beta(\frac{\mu^2}{k} - 3\rho_2) \;=\; 0.$
$X^2 = X_1 + \alpha X_2 + \beta X_4$ $\alpha, \beta \in \mathbb{R}.$	B	$(k - \alpha^2\rho_1)\, Z'' + k\, W' - \beta\rho_1 \;=\; 0,$ $(\chi'(W') - \alpha^2\rho_2)\, W'' - k\, Z' + \alpha d\, W' - k\, W = 0.$
$X^3 = X_2 + \alpha X_6 + \beta X_7 + \gamma X_8$ $\alpha \neq 0,\ \beta, \gamma \in \mathbb{R}.$	C	$\rho_1\, Z'' - k e^{-\frac{d}{2\rho_2}\zeta}\left(\beta\cos(\frac{\mu\zeta}{2\rho_2}) + \gamma\sin(\frac{\mu\zeta}{2\rho_2})\right) - \alpha d = 0,$ $\rho_2\, W'' + d\, W' + k\, W = 0.$
$X^4 = X_2 + \alpha X_5 + \beta X_7 + \gamma X_8$ $\alpha, \beta, \gamma \in \mathbb{R}.$	D	$\rho_1\, Z'' - k e^{-\frac{d}{2\rho_2}\zeta}\left(\beta\cos(\frac{\mu\zeta}{2\rho_2}) + \gamma\sin(\frac{\mu\zeta}{2\rho_2})\right) \;=\; 0,$ $\rho_2\, W'' + d\, W' + k\, W = 0.$

$A:$	$\phi(t,x) \;=\; Z(\zeta) - \frac{\beta}{6}t^2(\alpha t - 3x),$	$\psi(t,x) \;=\; W(\zeta) - \frac{\beta}{2}t(t - \frac{2d}{k}),$	$\zeta \;=\; x - \alpha t.$
$B:$	$\phi(t,x) \;=\; Z(\zeta) + \frac{\beta}{2}t^2,$	$\psi(t,x) \;=\; W(\zeta),$	$\zeta \;=\; x - \alpha t.$
$C:$	$\phi(t,x) \;=\; Z(\zeta) + \frac{\alpha}{2}tx^2,$	$\psi(t,x) \;=\; W(\zeta) + \alpha x(\frac{d}{k} - t) + x e^{-\frac{d}{2\rho_2}t}\left(\beta\cos(\frac{\mu t}{2\rho_2}) + \gamma\sin(\frac{\mu t}{2\rho_2})\right),$	$\zeta \;=\; t.$
$D:$	$\phi(t,x) \;=\; Z(\zeta) + \frac{\alpha}{2}x^2,$	$\psi(t,x) \;=\; W(\zeta) - x\,\alpha + x e^{-\frac{d}{2\rho_2}t}\left(\beta\cos(\frac{\mu t}{2\rho_2}) - \gamma\sin(\frac{\mu t}{2\rho_2})\right),$	$\zeta \;=\; t.$

4.5 Discussion and concluding remarks

The complete Lie symmetry classification of a non-linear Timoshenko system of PDEs with frictional damping term in rotational angle is performed. The classification is related to the arbitrary dependence on the rotation moment $\chi(\psi_x)$. A Lie symmetry analysis is performed in three cases for non-linear rotational moment. The three cases depend on the sign of the parameter $d^2 - 4k\rho_2$. The one-dimensional optimal system is derived for each one of the three cases. All possible invariant forms and their corresponding reductions for each vector field in the optimal systems are found. These reductions to systems of ODEs are given in Table 4.4, Table 4.5 and Table 4.6. They are described by optimal reduction where all non-similar invariant solutions under symmetry transformations can be given from the solution of these reduced system of ODEs.

CHAPTER 5

TIMOSHENKO BEAM WITH NON-LINEAR WEAK DAMPING

In this chapter we consider a non-linear class of Timoshenko system with frictional damping in rotation angle. The non-linearity is due to the arbitrary dependence on the rotational damping ψ_t. Lie symmetries and their Lie group transformations for Timoshenko system are presented. An optimal system of one-dimensional subalgebras of the corresponding Lie algebra is derived. All possible invariant variables of the optimal system are obtained. The corresponding reduced systems of ODE's are also provided. All possible non-similar invariant conditions prescribed on invariant surfaces under symmetry transformations are given. As an application, we study a beam which is hinged at one end where a constant control torque is applied, and free at the other end where the linear control force is applied.

5.1 Introduction

The problem of transverse vibration of beams is of importance in many engineering situations. Some of the early studies were based upon the Euler-bernoulli model which takes into account bending moment and the lateral displacement. Later models were based upon adding shear or rotary inertia effect. Timoshenko [79, 80] proposed taking into consideration the shear as well as the rotation effects which proved to be suitable for non-slender beams and high frequency vibrations. Mustafa and Messaoudi [47], and Fatiha [23] studied stability of the following Timoshenko system with non-linear frictional damping term in one equation

$$\begin{aligned} &\rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0, \\ &\rho_2\psi_{tt} - EI\psi_{xx} + k(\varphi_x + \psi) + \chi(\psi_t) = 0, \end{aligned} \tag{5.1.1}$$

where the functions φ, ψ depend on $(t, x) \in (0, \infty) \times (0, L)$ and model the vertical displacement of a beam, and the rotation angle of a filament, respectively. The shear angle is $\psi - \varphi_x$, and L denotes the length of the beam. The physical parameters appearing in the system are ρ_1, the mass density per unit length, ρ_2, the polar moment of inertia of a cross section, E, Young's modulus of elasticity, I, the moment of inertia of the cross section, and k, the shear modulus. In this chapter the beam is assumed to be uniform, that is the physical parameters ρ_1, ρ_2, EI and k are all positive constants, and χ is an arbitrary nonlinear function of the damping with respect to rotation angle ψ.

We have solved this system by determining its symmetries and performing a

Lie theoretic analysis as detailed below.

As mention in section (1.8) deriving the optimal system is one of the main applications of Lie group analysis to differential equations [53, 38, 51, 33]. Optimal system is an effective way of classify the group invariant solutions which in turn leads to non-similar invariant solutions under symmetry transformations.

Recall that one calls a list $\{\tilde{v}_\alpha\}_{\alpha \in \mathcal{A}}$ a one-dimensional optimal system, if it satisfies the conditions: (1) completeness, i.e. any one-dimensional subalgebra is equivalent to some $\tilde{v}_\alpha$; (2) inequivalence, i.e. $\tilde{v}_\alpha$ and $\tilde{v}_\beta$ are inequivalent for distinct α and β. In this paper, we have used all the basic invariants to determine the conjugacy classes of one-dimensional subalgebras, and at the same time shown that these representatives are comprehensive and mutually inequivalent. This is done by using the formula given in section 3 of [2]. This formula is more or less a direct consequence of definitions. For the convenience of the reader, a detailed explanation is given in section 5.2. The idea of using invariant functions to determine conjugacy classes is also discussed in [40, 32].

Here is a brief outline of the chapter. In section 5.2, a formula for computing invariants in the adjoint representation is given. In section 5.3, Lie symmetries and their Lie group transformations for Timoshenko system are presented. In section 5.4, an optimal system of one-dimensional subalgebras of the corresponding Lie algebra is derived. In section 5.5, all possible invariant variables of the optimal system are presented; moreover, the corresponding reduced systems of ODEs are also given. As an illustration, some invariant solutions are given explicitly by solving the reduced systems of ODEs. Also all possible non-similar invariant conditions prescribed on in invariant surfaces under symmetry transformations are given. A hinged-free beam has been considered in Example 5.5.1 with a constant torque control at the hinged end, and a linear force control at the free end.

5.2 Invariants in the adjoint representations

The following formula is stated without a proof in [2]. Let $X_1, ..., X_N$ be a basis of $\mathcal{L}$ and $\omega_1, ..., \omega_N$ be the corresponding dual basis of $\mathcal{L}^*$. Let X be an element of $\mathcal{L}$ and $Y = \sum_{i=1}^{N} x_i X_i$ be a general element of $\mathcal{L}$. Then $X_{\mathcal{L}}$-the fundamental vector field corresponding to X in the adjoint representation is given by

$$X_{\mathcal{L}}(Y) = \sum_{1 \leq i,j \leq N} x_i \omega_j([X, X_i]) \frac{\partial}{\partial x_j}.$$

To see this recall that a vector field V on an open set Ω of $\mathbb{R}^{\mathbb{N}}$ assigns to each point p of Ω of a vector $V(p) = (V_1(p), ..., V_N(p))$ therefore, if $e_1, ..., e_N$ is a basis of $\mathbb{R}^{\mathbb{N}}$ and $\omega_1, ..., \omega_N$ is the dual basis, then

$$V(p) = \sum_{i=1}^{N} \omega_i(V(p)) e_i.$$

The vector field V gives rise to the differential operator of taking the directional derivative of any function f in the direction of $V(p)$. We identify V with the operator $\sum_{j=1}^{N} \omega_j(V(p)) \frac{\partial}{\partial x_j} \mid_p$.

Specializing to the adjoint representation, the fundamental vector field $X_{\mathcal{L}}$ is given by

$$X_{\mathcal{L}}(Y) = \frac{\partial}{\partial t} \mid_{t=0} e^{tX} Y e^{-tX} = [X, Y].$$

Therefore if $Y = \sum_{i=1}^{N} x_i X_i$ then

$$X_{\mathcal{L}}(Y) = \sum_{1 \leq i,j \leq N} x_i w_j([X, X_i]) X_j = \sum_{1 \leq i,j \leq N} x_i w_j([X, X_i]) \frac{\partial}{\partial x_j}.$$

Application to computations of invariants.

Let f be a function on $\mathcal{L}$ which is invariant in the adjoint representation of $\mathcal{L}$. This means that $f(gYg^{-1}) = f(Y)$ for all g in the group generated by $\langle exp(tX); t \in \mathbb{R}, X \in \mathcal{L} \rangle$. Therefore if we write $f(x_1 X_1 + ... + x_n X_n) = f(x_1, ..., x_n)$, then $\sum_{1 \leq i,j \leq N} x_i \omega_j([X, X_i]) \frac{\partial}{\partial x_j} f = 0$. Letting X run through the basis $\{X_1, ..., X_N\}$ we get the system

$$\sum_{1 \leq i,j \leq N} x_i \omega_j([X_k, X_i]) \frac{\partial f}{\partial x_j} = 0, \quad k = 1, ..., N. \tag{5.2.1}$$

The system (5.2.1) needs to be solved only for a set of generators of the Lie algebra $\mathcal{L}$. The solution of this linear system of PDEs gives the invariants in the adjoint representations.

For a solvable Lie algebra $\mathcal{L}$, it is best to work with the central series for the commutator $\mathcal{L}'$ of $\mathcal{L}$, which is a nilpotent subalgebra. Therefore, one can find a chain of subalgebras, each an ideal in its successor and each of codimension 1.

Using this chain, one is essentially dealing with invariants of a single vector field, namely if we have found the basic invariants for a subalgebras S which is an ideal of codimension 1 in $\tilde{S}$, then $\tilde{S}\diagup S$ operates an invariants of S and this gives the invariants of $\tilde{S}$. Having obtained the invariants of $\mathcal{L}'$, one can again reach $\mathcal{L}$ by a series of ideals, each of codimension 1, to determine the basic invariants of $\mathcal{L}$. Such a series also gives a factorization of the group $G = \langle exp(Ad\ X) : X \in \mathcal{L}\rangle$ in terms of one dimensional subgroups.

The advantage of working with invariant functions is that if f is an invariant of $\mathcal{L}$ in the adjoint representation, then the adjoint group operates on the level sets as well as sub-level sets of f and one can systematically use invariants to work on level and sub-level sets of low dimensions to determine all the conjugacy classes of 1-dimensional subalgebras. This procedure will be used in section 5.4 to determine 1-dimensional optimal systems of the nonlinear damped Timpshenko system. This in turn will be used in section 5.5 to determine optimal reduction.

5.3 Lie's infinitesimal analysis

Lie symmetry method [5, 6, 35], which is invoked in the sequel to apply Lie symmetry analysis for Timoshenko system (5.1.1). Consider now the kth-order system of PDEs in n independent variables $x = (x^1, x^2, ..., x^n)$ and m dependent variables $u = (u^1, u^2, ..., u^m)$, viz.

$$E_\alpha(x, u, ..., u_{(k)}) = 0, \quad \alpha = 1, ..., m \ , \tag{5.3.1}$$

where $u_{(1)}, u_{(2)}, ..., u_{(k)}$ denote the collections of all first, second, ...,kth-order partial derivatives, i.e., $u_i^\alpha = D_i(u^\alpha)$, $u_{ij}^\alpha = D_j D_i(u^\alpha)$,...,respectively, in which the operator of total differentiation with respect to x^i is

$$D_i = \tfrac{\partial}{\partial x^i} + u_i^\alpha \tfrac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \tfrac{\partial}{\partial u_j^\alpha} + ..., \quad i = 1, ..., n \ , \tag{5.3.2}$$

with summation implied for repeated indices. *The Lie-Bäcklund operator* is

$$X = \xi^i \tfrac{\partial}{\partial x^i} + \eta^\alpha \tfrac{\partial}{\partial u^\alpha} \quad \xi^i, \eta^\alpha \in \mathcal{A} \ , \tag{5.3.3}$$

where $\mathcal{A}$ is the space of *differential functions.* The operator (5.3.3) is an abbreviated form of the infinite formal sum

$$\begin{aligned} X^{(s)} &= \xi^i \tfrac{\partial}{\partial x^i} + \eta^\alpha \tfrac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} \zeta_{i_1 i_2 \dots i_s}^\alpha \tfrac{\partial}{\partial u_{i_1 i_2 \dots i_s}^\alpha}, \\ &= \xi^i D_i + W^\alpha \tfrac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} D_{i_1} ... D_{i_s}(W^\alpha) \tfrac{\partial}{\partial u_{i_1 i_2 \dots i_s}^\alpha}, \end{aligned} \tag{5.3.4}$$

where the additional coefficients are determined uniquely by the prolongation formulae

$$\begin{aligned} &\zeta_i^\alpha = D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j) = D_i(W^\alpha) + \xi^j u_{ij}^\alpha, \\ &\zeta_{i_1 \dots i_s}^\alpha = D_{i_s}(\zeta_{i_1 \dots i_{s-1}}^\alpha) - u_{j i_1 \dots i_{s-1}}^\alpha D_{i_s}(\xi^j) = D_{i_1} ... D_{i_s}(W^\alpha) + \xi^j u_{j i_1 \dots i_s}^\alpha, \quad s > 1, \end{aligned} \tag{5.3.5}$$

in which W^α is the *Lie characteristic function*

$$W^\alpha = \eta^\alpha - \xi^j u_j^\alpha. \tag{5.3.6}$$

Using the invariance condition of the system of PDEs (5.1.1)

$$
\begin{aligned}
&X^{[2]}\left(\rho_1\varphi_{tt} - k(\varphi_x + \psi)_x\right)|_{(5.1.1)} = 0, \\
&X^{[2]}\left(\rho_2\psi_{tt} - EI\psi_{xx} + k(\varphi_x + \psi) + \chi(\psi_t)\right)|_{(5.1.1)} = 0,
\end{aligned} \tag{5.3.7}
$$

and comparing coefficients of the various derivatives of the dependent variables φ and ψ yield an over-determined linear PDE system. Carrying out the Janet basis of this over-determined system in the degree reverse lexicographical ordering as $\psi > \phi > x > t$ and $\eta_2 > \eta_1 > \xi_2 > \xi_1$ by using the command "JanetBasis" involved in the Maple package "Janet" [4], leads to the following determining equations:

$$
\begin{aligned}
&\big[\ \eta^1_\psi,\ \xi^2_\psi,\ \xi^1_\psi,\ \eta^2_\varphi,\ \eta^1_\varphi - \eta^2_\psi,\ \xi^2_\varphi,\ \xi^1_\varphi,\ \xi^2_x,\ \xi^1_x,\ \xi^2_t,\ \xi^1_t,\ \eta^2_{\psi,\psi},\ \eta^2_{\psi,x},\ \eta^2_{\psi,t},\ \eta^1_{\psi,t},\ \eta^2_{\varphi,x}, \\
&\eta^2_{\varphi,t},\ \eta^1_{\varphi,t},\ k\eta^1_x + k\eta^2 + (\psi_t\chi' - k\psi - \chi)\,\eta^2_\psi + \eta^2_t\chi' - EI\eta^2_{x,x} + \rho_2\eta^2_{t,t}, \\
&\eta^1_{x,x} - \tfrac{\rho_1}{k}\eta^1_{t,t} + \eta^2_x\ \big]
\end{aligned} \tag{5.3.8}
$$

The command "Denominators" involved in the Maple package "Janet" returns the functions by which the Janet basis algorithm had to divide. These functions may give rise new cases. The command "Denominators" tells us that the Janet basis of determining equations (5.3.8) is produced when $k \neq \frac{\rho_1 EI}{\rho_2}$. However, the Janet basis of determining equations for this case is equivalent to (5.3.8).

The solution of system (5.3.8) of determining equations is

$$
\xi^1 = c_1, \quad \xi^2 = c_2, \quad \eta^1 = c_0\ \varphi + f(t,x), \quad \eta^2 = c_0\ \psi + g(t,x) \tag{5.3.9}
$$

where $f(t,x)$ and $g(t,x)$ satisfy the following system

$$\begin{aligned}
\rho_1 f_{tt} - k f_{xx} - k g_x &= 0, \\
k f_x + k g + g_t \chi'(\psi_t) + \rho_2 g_{tt} - EI g_{xx} &= c_0 \left(\chi(\psi_t) - \psi_t \chi'(\psi_t)\right).
\end{aligned} \tag{5.3.10}$$

Differentiating the second equation of system (5.3.10) with respect to ψ_t gives

$$(g_t + c_0\ \psi_t)\,\chi''(\psi_t) = 0. \tag{5.3.11}$$

Since we are concerned with non-linear damping term then $\chi''(\psi_t) \neq 0$, which implies $g = g(x)$ and $c_0 = 0$. Solving system (5.3.10) yields

$$\begin{aligned}
f(t,x) &= c_3 + c_4 t - c_5 x - \tfrac{1}{2} c_6 x^2 + \left(-\tfrac{1}{6} x^3 + \tfrac{EI}{k} x\right) c_7 \\
&\quad + \left(-\tfrac{1}{24} x^4 + \tfrac{EI}{2k} x^2 + \tfrac{EI}{2\rho_1} t^2\right) c_8, \\
g(x) &= c_5 + c_6 x + \tfrac{1}{2} c_7 x^2 + \tfrac{1}{6} c_8 x^3.
\end{aligned} \tag{5.3.12}$$

The Lie point symmetry generators admitted by the system (5.1.1) are given by

$$\begin{array}{l|l}
X_1 = \frac{\partial}{\partial t}, & X_5 = -x\frac{\partial}{\partial \varphi} + \frac{\partial}{\partial \psi}, \\
X_2 = \frac{\partial}{\partial x}, & X_6 = -\frac{1}{2}x^2\frac{\partial}{\partial \varphi} + x\frac{\partial}{\partial \psi}, \\
X_3 = \frac{\partial}{\partial \varphi}, & X_7 = (-\frac{1}{6}x^3 + \frac{EI}{k}x)\frac{\partial}{\partial \varphi} + \frac{1}{2}x^2\frac{\partial}{\partial \psi}, \\
X_4 = t\frac{\partial}{\partial \varphi}, & X_8 = (-\frac{1}{24}x^4 + \frac{EI}{2k}x^2 + \frac{EI}{2\rho_1}t^2)\frac{\partial}{\partial \varphi} + \frac{1}{6}x^3\frac{\partial}{\partial \psi}.
\end{array} \tag{5.3.13}$$

In order to obtain the group transformations which are generated by the resulting infinitesimal symmetry generators (5.3.13), we need to solve the following initial value problem:

$$\begin{aligned}
\frac{d\tilde{t}(\epsilon)}{d\epsilon} &= \xi^1\left(\tilde{t}(\epsilon),\tilde{x}(\epsilon),\tilde{\varphi}(\epsilon),\tilde{\psi}(\epsilon)\right), \quad \tilde{t}(0)=t,\\
\frac{d\tilde{x}(\epsilon)}{d\epsilon} &= \xi^2\left(\tilde{t}(\epsilon),\tilde{x}(\epsilon),\tilde{\varphi}(\epsilon),\tilde{\psi}(\epsilon)\right), \quad \tilde{x}(0)=x,\\
\frac{d\tilde{\varphi}(\epsilon)}{d\epsilon} &= \eta^1\left(\tilde{t}(\epsilon),\tilde{x}(\epsilon),\tilde{\varphi}(\epsilon),\tilde{\psi}(\epsilon)\right), \quad \tilde{\varphi}(0)=\varphi,\\
\frac{d\tilde{\psi}(\epsilon)}{d\epsilon} &= \eta^2\left(\tilde{t}(\epsilon),\tilde{x}(\epsilon),\tilde{\varphi}(\epsilon),\tilde{\psi}(\epsilon)\right), \quad \tilde{\psi}(0)=\psi.
\end{aligned} \tag{5.3.14}$$

The one parameter group $G_i(\epsilon)=e^{\epsilon X_i}$ generated by X_i for $i=1,...,8$, are as follows:

$$\begin{aligned}
&G_1(\epsilon):(t,x,\varphi,\psi)\mapsto(t+\epsilon,x,\varphi,\psi),\\
&G_2(\epsilon):(t,x,\varphi,\psi)\mapsto(t,x+\epsilon,\varphi,\psi)\\
&G_3(\epsilon):(t,x,\varphi,\psi)\mapsto(t,x,\varphi+\epsilon,\psi),\\
&G_4(\epsilon):(t,x,\varphi,\psi)\mapsto(t,x,\varphi+\epsilon\ t,\psi),\\
&G_5(\epsilon):(t,x,\varphi,\psi)\mapsto(t,x,\varphi-\epsilon\ x,\psi+\epsilon),\\
&G_6(\epsilon):(t,x,\varphi,\psi)\mapsto(t,x,\varphi-\tfrac{1}{2}\ \epsilon\ x^2,\psi+\epsilon\ x),\\
&G_7(\epsilon):(t,x,\varphi,\psi)\mapsto(t,x,\varphi-\tfrac{1}{6}\epsilon x^3+\tfrac{EI}{k}\epsilon x,\psi+\tfrac{1}{2}\epsilon x^2),\\
&G_8(\epsilon):(t,x,\varphi,\psi)\mapsto(t,x,\varphi-\tfrac{1}{24}\epsilon x^4+\tfrac{EI}{2k}\epsilon x^2+\tfrac{EI}{2\rho_1}\epsilon t^2,\psi+\tfrac{1}{6}\epsilon x^3).
\end{aligned} \tag{5.3.15}$$

Theorem 5.3.1 *If $\varphi=f(t,x)$ and $\psi=g(t,x)$ is a solution of the Timoshenko* *system (5.1.1), then so is*

$$\begin{aligned}\varphi = & \; f\left(t+\epsilon_1, x+\epsilon_2\right)+\epsilon_3+\epsilon_4\left(t+\epsilon_1\right)-\epsilon_5\left(x+\epsilon_2\right)-\epsilon_6(\epsilon_2 x+\frac{1}{2}\,\epsilon_2^2+\frac{1}{2}\,x^2) \\ & -\epsilon_7(\frac{1}{2}\,x\;\epsilon_2^2-(\frac{EI}{k}-\frac{1}{2}\,x^2)\epsilon_2+\frac{1}{6}\,x^3-\frac{EI}{k}x+\frac{1}{6}\,\epsilon_2^3)-\epsilon_8\left(\frac{1}{6}\,x\;\epsilon_2^3-\frac{EI}{2k}x^2\right. \\ & +\epsilon_2^2(\frac{1}{4}\,x^2-\frac{EI}{2k})+\frac{1}{24}\,x^4-\epsilon_2(\frac{EI}{k}x-\frac{1}{6}\,x^3)-\frac{EI}{\rho_1}\epsilon_1 t+\frac{1}{24}\,\epsilon_2^4 \\ & \left.-\frac{EI}{2\rho_1}\epsilon_1^2-\frac{EI}{2\rho_1}t^2\right), \\ \psi = & \; g\left(t+\epsilon_1, x+\epsilon_2\right)+\epsilon_5+\epsilon_6\left(x+\epsilon_2\right)+\frac{1}{2}\,\epsilon_7\left(x+\epsilon_2\right)^2+\frac{1}{6}\,\epsilon_8\left(x+\epsilon_2\right)^3,\end{aligned} \tag{5.3.16}$$

where $\{\epsilon_i\}_{i=1}^{8}$ *are arbitrary real numbers.*

Proof. *The eight parameters group*

$$G(\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4,\epsilon_5,\epsilon_6,\epsilon_7,\epsilon_8)=G_8(\epsilon_8)\circ G_7(\epsilon_7)\circ \quad ... \quad \circ G_1(\epsilon_1),$$

generated by X_i *for* $i=1,...,8,$ *can be given by the composition of the transformations (5.3.15) as follows:*

$$\begin{aligned}& G:(t,x,\varphi,\psi)\longmapsto(t+\epsilon_1,\quad x+\epsilon_2,\quad \phi+\epsilon_3+\epsilon_4(t+\epsilon_1)-\epsilon_5(x+\epsilon_2) \\ & -\epsilon_6(\epsilon_2 x+\tfrac{1}{2}\,\epsilon_2^2+\tfrac{1}{2}\,x^2)-\epsilon_7(\tfrac{1}{2}\,x\epsilon_2^2-(\tfrac{EI}{k}-\tfrac{1}{2}\,x^2)\epsilon_2+\tfrac{1}{6}\,x^3-\tfrac{EI}{k}x+\tfrac{1}{6}\,\epsilon_2^3) \\ & -\epsilon_8(\tfrac{1}{6}\,x\epsilon_2^3-\tfrac{EI}{2k}x^2+(\tfrac{1}{4}\,x^2-\tfrac{EI}{2k})\epsilon_2^2+\tfrac{1}{24}\,x^4-(\tfrac{EI}{k}x-\tfrac{1}{6}\,x^3)\epsilon_2-\tfrac{EI}{\rho_1}\epsilon_1 t+\tfrac{1}{24}\,\epsilon_2^4 \\ & -\tfrac{EI}{2\rho_1}\epsilon_1^2-\tfrac{EI}{2\rho_1}t^2),\quad \psi+\epsilon_5+\epsilon_6(x+\epsilon_2)+\tfrac{1}{2}\,\epsilon_7(x+\epsilon_2)^2+\tfrac{1}{6}\,\epsilon_8(x+\epsilon_2)^3).\end{aligned} \tag{5.3.17}$$

∎

5.4 One-dimensional optimal system

In this section, we give the one-dimensional optimal system for the algebra with basis (5.3.13). In order to find the optimal system, one needs to classify the one-dimensional sub-algebras under the action of the adjoint representation. We follow the algorithm explained by Ibragimov [38] and Olver [51].

The non-zero commutators of the Lie algebra $\mathcal{L}^8$ with basis (5.3.13) are given by

$$\begin{aligned} &[X_1, X_4] = X_3, \quad [X_1, X_8] = \tfrac{EI}{\rho_1} X_4 \ , \qquad [X_2, X_5] = -X_3, \\ &[X_2, X_6] = X_5, \quad [X_2, X_7] = X_6 + \tfrac{EI}{k} X_3, \quad [X_2, X_8] = X_7. \end{aligned} \tag{5.4.1}$$

Recall that the adjoint representation is given by

$$Ad(exp(\epsilon X_i).X_j) = X_j - \epsilon[X_i, X_j] + \tfrac{\epsilon^2}{2!}[X_i, [X_i, X_j]] - \tfrac{\epsilon^3}{3!}[X_i, [X_i, X_j]] + ...,$$

The Lie algebra $\mathcal{L}^8$ is solvable and the adjoint table is given in Table 5.1 below:

Table 5.1: Adjoint table

$Ad(e^{\epsilon})$	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
X_1	X_1	X_2	X_3	$X_4-\epsilon X_3$	X_5	X_6	X_7	Y_1
X_2	X_1	X_2	X_3	X_4	$X_5+\epsilon X_3$	Y_2	Y_3	Y_4
X_3	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
X_4	$X_1+\epsilon X_3$	X_2	X_3	X_4	X_5	X_6	X_7	X_8
X_5	X_1	$X_2-\epsilon X_3$	X_3	X_4	X_5	X_6	X_7	X_8
X_6	X_1	$X_2+\epsilon X_5$	X_3	X_4	X_5	X_6	X_7	X_8
X_7	X_1	$X_2+\epsilon(X_6+\frac{EI}{k}X_3)$	X_3	X_4	X_5	X_6	X_7	X_8
X_8	$X_1+\frac{EI}{k}\epsilon X_4$	$X_2+\epsilon X_7$	X_3	X_4	X_5	X_6	X_7	X_8

$Y_1 = X_8 - \frac{EI}{\rho_1}\epsilon X_4 + \frac{EI}{2\rho_1}\epsilon^2 X_3,$ $\quad Y_2 = X_6 - \epsilon X_5 - \frac{1}{2}\epsilon^2 X_3,$

$Y_3 = X_7 - \epsilon(X_6 + \frac{EI}{k}X_3) + \frac{\epsilon^2}{2}X_5 + \frac{\epsilon^3}{6}X_3,$ $\quad Y_4 = X_8 - \epsilon X_7 + \frac{\epsilon^2}{2}(X_6 + \frac{EI}{k}X_3) - \frac{\epsilon^3}{6}X_5 - \frac{\epsilon^4}{24}X_3.$

The adjoint group is group generated by $\langle exp(adX) : X \in \mathcal{L}\rangle$. Using the solvability of $\mathcal{L}$, this group consists of the elements

$$A = Ad(e^{\epsilon_1 X_1}).Ad(e^{\epsilon_2 X_2}). \dots .Ad(e^{\epsilon_8 X_8}).$$

Therefore, A is given by

$$A = \begin{pmatrix} 1 & 0 & \epsilon_4 & \frac{EI}{\rho_1}\epsilon_8 & 0 & 0 & 0 & 0 \\ 0 & 1 & \frac{EI}{k}\epsilon_7 - \epsilon_5 & 0 & \epsilon_6 & \epsilon_7 & \epsilon_8 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\epsilon_1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \epsilon_2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}\epsilon_2^2 & 0 & -\epsilon_2 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{6k}\epsilon_2(6EI - k\epsilon_2^2) & 0 & \frac{1}{2}\epsilon_2^2 & -\epsilon_2 & 1 & 0 \\ 0 & 0 & \frac{EI}{2\rho_1}\epsilon_1^2 + \frac{1}{24k}\epsilon_2^2(12EI - k\epsilon_2^2) & -\frac{EI}{\rho_1}\epsilon_1 & -\frac{1}{6}\epsilon_2^3 & \frac{1}{2}\epsilon_2^2 & -\epsilon_2 & 1 \end{pmatrix}. \tag{5.4.2}$$

Theorem 5.4.1 *An optimal system of one-dimensional sub-algebras of $\mathcal{L}^8$ with basis (5.3.13) is provided by the following generators*

$$\begin{array}{ll|ll} X^1 = X_1 + \alpha X_2 + \beta X_8, & \alpha, \beta \neq 0, & X^9 = \alpha X_3 + \beta X_5 + \gamma X_6 + X_8, & \alpha, \beta, \gamma \in \mathbb{R}, \\ X^2 = X_1 + \alpha X_2 + \beta X_4, & \alpha \neq 0, \beta \in \mathbb{R}, & X^{10} = X_4 + \alpha X_5 + \beta X_7, & \alpha \in \mathbb{R}, \beta \neq 0, \\ X^3 = X_1 + \alpha X_5 + \beta X_6 + \gamma X_8, & \alpha, \beta \in \mathbb{R}, \gamma \neq 0, & X^{11} = X_4 + \alpha X_6, & \alpha \neq 0, \\ X^4 = X_1 + \alpha X_5 + \beta X_7, & \alpha \in \mathbb{R}, \beta \neq 0, & X^{12} = X_4 + \alpha X_5, & \alpha \in \mathbb{R}, \\ X^5 = X_1 + \alpha X_6, & \alpha \neq 0, & X^{13} = \alpha X_3 + \beta X_5 + X_7, & \alpha, \beta \in \mathbb{R}, \\ X^6 = X_1 + \alpha X_5, & \alpha \in \mathbb{R}, & X^{14} = \alpha X_3 + X_7, & \alpha \in \mathbb{R}, \\ X^7 = X_2 + \alpha X_8, & \alpha \neq 0, & X^{15} = X_5, & \\ X^8 = X_2 + \alpha X_4, & \alpha \in \mathbb{R}, & X^{16} = X_3. & \end{array} \tag{5.4.3}$$

Proof. *Let X and $\tilde{X}$ be two elements in the Lie algebra $\mathcal{L}^8$ with basis (5.3.13) given by $X = \sum_{i=1}^{8} a_i X_i$ and $\tilde{X} = \sum_{i=1}^{8} \tilde{a}_i X_i$. For simplicity, we will write X and $\tilde{X}$ as row vectors of the coefficients on the form $X = (\, a_1 \; a_2 \; . \; . \; . \; a_8 \,)$ and $\tilde{X} = (\, \tilde{a}_1 \; \tilde{a}_2 \; . \; . \; . \; \tilde{a}_8 \,)$. Then in order for X and $\tilde{X}$ to be in the same conjugacy* *class, we must have $\tilde{X} = XA$, where A is given by (5.4.2). So, the theorem is proved by solving the system*

$$
\begin{aligned}
\tilde{a}_1 &= a_1, \\
\tilde{a}_2 &= a_2, \\
\tilde{a}_3 &= a_1\,\epsilon_4 + \tfrac{EI}{k}\,a_2\epsilon_7 - a_2\,\epsilon_5 + a_3 - a_4\,\epsilon_1 + a_5\,\epsilon_2 - \tfrac{1}{2}\,a_6\,\epsilon_2^2 \\
&\quad + \tfrac{1}{6}\,a_7\,\epsilon_2^3 - \tfrac{EI}{k}a_7\;\epsilon_2 - \tfrac{1}{24}\,a_8\,\epsilon_2^4 + \tfrac{1}{2}\,\tfrac{EI}{\rho_1}a_8\;\epsilon_1^2 + \tfrac{1}{2}\,\tfrac{EI}{k}a_8\;\epsilon_2^2, \\
\tilde{a}_4 &= \tfrac{EI}{\rho_1}a_1\epsilon_8 - \tfrac{EI}{\rho_1}a_8\epsilon_1 + a_4, \\
\tilde{a}_5 &= a_2\,\epsilon_6 + a_5 - a_6\,\epsilon_2 + \tfrac{1}{2}\,a_7\,\epsilon_2^2 - \tfrac{1}{6}\,a_8\,\epsilon_2^3, \\
\tilde{a}_6 &= a_2\,\epsilon_7 + a_6 - a_7\,\epsilon_2 + \tfrac{1}{2}\,a_8\,\epsilon_2^2, \\
\tilde{a}_7 &= a_2\epsilon_8 - a_8\epsilon_2 + a_7, \\
\tilde{a}_8 &= a_8,
\end{aligned}
\tag{5.4.4}
$$

for $\{\epsilon_i\}_{i=1}^8$ in term of $\{a_i\}_{i=1}^8$ in order to get the simplest values of $\{\tilde{a}_i\}_{i=1}^8$. The results are presented for different cases in a tree diagram where each of its vertices is an invariant function, and its leafs are given completely. Moreover, one can verify that all the parameters α, β and γ appearing in each case are invariants. So, the inequivalence and completeness conditions are satisfied.

All joint invariants are obtained by solving the following system of PDEs which *is given by using the formula (5.2.1):*

$$\begin{cases} a_4\frac{\partial\Phi}{\partial a_3} + \frac{EI}{\rho_1}a_8\frac{\partial\Phi}{\partial a_4} = 0, \\ a_5\frac{\partial\Phi}{\partial a_3} - \frac{EI}{k}a_7\frac{\partial\Phi}{\partial a_3} - a_6\frac{\partial\Phi}{\partial a_5} - a_7\frac{\partial\Phi}{\partial a_6} - a_8\frac{\partial\Phi}{\partial a_7} = 0, \\ a_1\frac{\partial\Phi}{\partial a_3} = 0, \\ a_2\frac{\partial\Phi}{\partial a_3} = 0, \\ a_2\frac{\partial\Phi}{\partial a_5} = 0, \\ a_2\frac{\partial\Phi}{\partial a_6} + \frac{EI}{k}a_2\frac{\partial\Phi}{\partial a_3} = 0, \\ a_2\frac{\partial\Phi}{\partial a_7} + \frac{EI}{\rho_1}a_1\frac{\partial\Phi}{\partial a_4} = 0. \end{cases} \tag{5.4.5}$$

By solving this system, we obtain $\Phi(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8) = F(a_1, a_2, a_8)$, *where* F *is an arbitrary function of* a_1, a_2, a_8*. Hence the basic invariants of Timoshenko system (5.1.1) are* a_1, a_2 *and* a_8*: this means that the first three vertices of the tree will be these invariants in any order. In our case we will consider the order* a_1, a_2, a_8*. As it is illustrated in the following tree diagram*

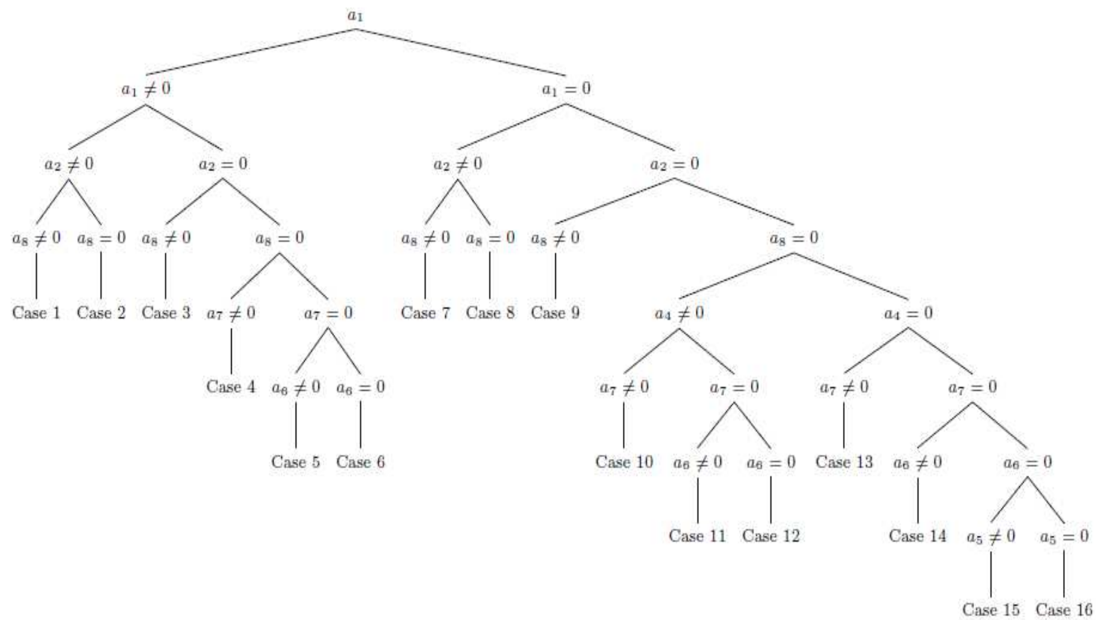


Figure 5.1: Tree diagram

The full details for each leaf are given as follows:

Case1: *$a_1 \neq 0,\ a_2 \neq 0,\ a_8 \neq 0$: Let $\epsilon_2 = \epsilon_3 = \epsilon_5 = 0$, $\epsilon_1 = \frac{\rho_1 a_2 a_4 - EIa_7}{EIa_2 a_8}$, $\epsilon_4 = \frac{\rho_1}{EI}\frac{a_4^2}{a_8} + \frac{EI}{k}a_6 - a_3 - \frac{EI}{2\rho_1}\frac{a_7^2}{a_2^2 a_8}$, $\epsilon_6 = -\frac{a_5}{a_2}$, $\epsilon_7 = -\frac{a_6}{a_2}$ and $\epsilon_8 = -\frac{a_7}{a_2}$ to have $\tilde{a_3} = \tilde{a_4} = \tilde{a_5} = \tilde{a_6} = \tilde{a_7} = 0$: the conjugacy class is $\langle X_1 + \alpha X_2 + \beta X_8\rangle$, with $\alpha, \beta \neq 0$.*

Case2: *$a_1 \neq 0,\ a_2 \neq 0,\ a_8 = 0$: Let $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_5 = 0$, $\epsilon_4 = \frac{EI}{k}a_6 - a_3$, $\epsilon_6 = -\frac{a_5}{a_2}$, $\epsilon_7 = -\frac{a_6}{a_2}$, $\epsilon_8 = -\frac{a_7}{a_2}$ to have $\tilde{a_3} = \tilde{a_5} = \tilde{a_6} = \tilde{a_7} = 0$. The conjugacy class is $\langle X_1 + \alpha X_2 + \beta X_4\rangle$, with $\alpha \neq 0$, $\beta \in \mathbb{R}$.*

Case3: *$a_1 \neq 0,\ a_2 = 0,\ a_8 \neq 0$: Let $\epsilon_1 = 0$, $\epsilon_2 = \frac{a_7}{a_8}$, $\epsilon_4 = \frac{EI}{2k}\frac{a_7^2}{a_8} - a_3 - \frac{a_5 a_7}{a_8} + \frac{a_6 a_7^2}{2a_8^2} - \frac{a_7^4}{8a_8^3}$ and $\epsilon_8 = -\frac{\rho_1}{EI}a_4$ to make $\tilde{a_3} = \tilde{a_4} = \tilde{a_7} = 0$: the conjugacy class is $\langle X_1 + \alpha X_5 + \beta X_6 + \gamma X_8\rangle$, $\alpha, \beta \in \mathbb{R}$, $\gamma \neq 0$.*

When $a_1 \neq 0$, $a_2 = 0$, $a_8 = 0$, we need to solve system (5.4.5) again to see what are the invariants as well the next vertices of the tree.

Solving system (5.4.5) taking into account that $a_2 = a_8 = 0$ gives $\Phi(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8) = F(a_1, a_7, -2a_5a_7 + a_6^2)$, so we can consider a_7 to be the next vertex of the tree.

Case4: *$a_1 \neq 0$, $a_2 = 0$, $a_8 = 0$, $a_7 \neq 0$: Let $\epsilon_1 = 0$, $\epsilon_2 = \frac{a_6}{a_7}$, $\epsilon_4 = \frac{EI}{k}a_6 - a_3 - \frac{a_5a_6}{a_7} + \frac{a_6^2}{3a_7^2}$ and $\epsilon_8 = -\frac{\rho_1}{EI}a_4$ to make $\tilde{a}_3 = \tilde{a}_4 = \tilde{a}_6 = \tilde{a}_8 = 0$: the conjugacy class is of the form $\langle X_1 + \alpha X_5 + \beta X_7 \rangle$, $\alpha \in \mathbb{R}, \beta \neq 0$.*

Again, for the case $a_1 \neq 0$, $a_2 = 0$, $a_8 = 0$, $a_7 = 0$, we resolve system (5.4.5) to have

$\Phi(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8) = F(a_1, a_6)$, so next vertex can be a_6:

Case5: *$a_1 \neq 0$, $a_2 = 0$, $a_8 = 0$, $a_7 = 0$, $a_6 \neq 0$: Let $\epsilon_1 = 0$, $\epsilon_2 = \frac{a_5}{a_6}$, $\epsilon_4 = -a_3 - \frac{a_5^2}{2a_6}$ and $\epsilon_8 = -\frac{\rho_1}{EI}a_4$ to make $\tilde{a}_3 = \tilde{a}_4 = \tilde{a}_5 = 0$: the corresponding conjugacy class is $\langle X_1 + \alpha X_6 \rangle$, where $\alpha \neq 0$.*

Case6: *$a_1 \neq 0$, $a_2 = 0$, $a_8 = 0$, $a_7 = 0$, $a_6 = 0$: Let $\epsilon_1 = \epsilon_2 = 0$, $\epsilon_4 = -a_3$ and $\epsilon_8 = -\frac{\rho_1}{EI}a_4$ to make $\tilde{a}_3 = \tilde{a}_4 = 0$: the conjugacy class is $\langle X_1 + \alpha X_5 \rangle$, $\alpha \in \mathbb{R}$.*

Case7: *$a_1 = 0$, $a_2 \neq 0$, $a_8 \neq 0$: Let $\epsilon_1 = \frac{\rho_1}{EI}\frac{a_4}{a_8}$, $\epsilon_2 = 0$, $\epsilon_5 = a_3 - \frac{EI}{k}a_6 - \frac{\rho_1}{2EI}\frac{a_4^2}{a_8}$, $\epsilon_6 = -a_5$, $\epsilon_7 = -a_6$ and $\epsilon_8 = -a_7$ to make $\tilde{a}_3 = \tilde{a}_4 = \tilde{a}_5 = \tilde{a}_6 = \tilde{a}_7 = 0$: the conjugacy class is $\langle X_2 + \alpha X_8 \rangle$, $\alpha \neq 0$.*

Case8: *$a_1 = 0$, $a_2 \neq 0$, $a_8 = 0$: Let $\epsilon_2 = 0$, $\epsilon_5 = a_3 - \epsilon_1 a_4 - \frac{EI}{k}a_6$, $\epsilon_6 = -a_5$, $\epsilon_7 = -a_6$ and $\epsilon_8 = -a_7$ to make $\tilde{a}_3 = \tilde{a}_5 = \tilde{a}_6 = \tilde{a}_6 = \tilde{a}_7 = 0$: the conjugacy class is $\langle X_2 + \alpha X_4 \rangle$, $\alpha \in \mathbb{R}$.*

Case9: *$a_1 = 0$, $a_2 = 0$, $a_8 \neq 0$: Let $\epsilon_1 = \frac{\rho_1}{EI}a_4$ and $\epsilon_2 = a_7$, to get $\tilde{a}_4 = \tilde{a}_7 = 0$: the conjugacy class is of the form $\langle \alpha X_3 + \beta X_5 + \gamma X_6 + X_8 \rangle$, $\alpha, \beta, \gamma \in \mathbb{R}$.*

By substituting $a_1 = a_2 = a_8 = 0$ in system (5.4.5) and solving it, we get $\Phi(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8) = F(a_4, a_7, -2a_5a_7 + a_6^2)$. So the next two vertices can be a_4 and a_7 in any order. We consider the order a_4, a_7.

Case10: *$a_1 = 0$, $a_2 = 0$, $a_8 = 0$, $a_4 \neq 0$, $a_7 \neq 0$: Let $\epsilon_1 = -\frac{EI}{k}a_6 + a_3 + \frac{a_5a_6}{a_7} - \frac{a_6^3}{3a_7^2}$ and $\epsilon_2 = \frac{a_6}{a_7}$ to make $\tilde{a_3} = \tilde{a_6} = 0$: the conjugacy class is $\langle X_4 + \alpha X_5 + \beta X_7 \rangle$, $\alpha \in \mathbb{R}, \beta \neq 0$.*

In case $a_1 = a_2 = a_8 = a_7 = 0$, solving system (5.4.5) yields $\Phi(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8) = F(a_4, a_6)$ which implies, after considering a_4 and a_7 as vertices, that one can consider the invariant a_6 as a new vertex.

Case11: *$a_1 = 0$, $a_2 = 0$, $a_8 = 0$, $a_4 \neq 0$, $a_7 = 0$, $a_6 \neq 0$: Let $\epsilon_1 = \frac{a_5^2}{2a_6} + a_3$ and $\epsilon_2 = \frac{a_5}{a_6}$ to make $\tilde{a_3} = \tilde{a_5} = 0$: the conjugacy class is $\langle X_4 + \alpha X_6 \rangle$, with $\alpha \neq 0$.*

Case12: *$a_1 = 0$, $a_2 = 0$, $a_8 = 0$, $a_4 \neq 0$, $a_7 = 0$, $a_6 = 0$: Let $\epsilon_1 = a_3$, $\epsilon_2 = 0$ to make $\tilde{a_3} = 0$: then the conjugacy class is $\langle X_4 + \alpha X_5 \rangle$, $\alpha \in \mathbb{R}$.*

Case13: *$a_1 = 0$, $a_2 = 0$, $a_8 = 0$, $a_4 = 0$, $a_7 \neq 0$: Let $\epsilon_2 = a_6$ to have $\tilde{a_6} = 0$: the conjugacy class is $\langle \alpha X_3 + \beta X_5 + X_7 \rangle$, $\alpha, \beta \in \mathbb{R}$.*

Case14: *$a_1 = 0$, $a_2 = 0$, $a_8 = 0$, $a_4 = 0$, $a_7 = 0$, $a_6 \neq 0$: Let $\epsilon_2 = a_5$ to have $\tilde{a_5} = 0$: the conjugacy class is $\langle \alpha X_3 + X_6 \rangle$, $\alpha \in \mathbb{R}$.*

When $a_6 = 0$ with $a_1 = a_2 = a_8 = a_4 = a_7 = 0$, solving the PDEs system (5.4.5) gives

$\Phi(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8) = F(a_5)$. So the next invariant vertex is a_5.

Case15: *$a_1 = 0$, $a_2 = 0$, $a_8 = 0$, $a_4 = 0$, $a_7 = 0$, $a_6 = 0$, $a_5 \neq 0$: Let $\epsilon_2 = -a_3$ to have $\tilde{a_3} = 0$: the conjugacy class is $\langle X_5 \rangle$.*

Case16: *$a_1 = 0$, $a_2 = 0$, $a_8 = 0$, $a_4 = 0$, $a_7 = 0$, $a_6 = 0$, $a_5 = 0$: the conjugacy class is $\langle X_3 \rangle$.*

$\blacksquare$

5.5 Optimal reductions and invariant solutions

It is known that the invariant solutions for PDEs can be determined by two procedures, namely the invariant form method and the direct substitution method [5]. The idea of looking for group invariant solutions generalizes quite naturally to PDEs with any number of independent and dependent variables. A one parameter group that acts nontrivially on one or more independent variables can be used to reduce the number of independent variables by one.

In this section, we focus on the invariant form method which requires that at least one of the infinitesimals ξ^1 and ξ^2 does not equal zero [33, 5]. Hence, we solve the invariance surface conditions explicitly by solving the corresponding characteristic equation given by

$$\frac{dt}{\xi^1(t,x,\varphi,\psi)} = \frac{dx}{\xi^2(t,x,\varphi,\psi)} = \frac{d\varphi}{\eta^1(t,x,\varphi,\psi)} = \frac{d\psi}{\eta^2(t,x,\varphi,\psi)}, \tag{5.5.1}$$

to get the corresponding invariants which are used to reduce the number of independent variables by one. The procedure is explained in details in the following example. Moreover, all possible invariant variables of the optimal system (5.4.3) and their corresponding reductions are given in Table 5.2.

As another application for the optimal system based on the definition of the invariant boundary value problem given in [5, 38], we classify the non-similar invariant conditions prescribed on invariant surfaces under symmetry transformations for Timoshenko system (5.1.1) as given in Table 5.3. This is achieved by finding invariant conditions as arbitrary functions of the invariants up to the first order, and invariant surfaces as arbitrary functions of the invariants of order zero which depend on the independent variables for each generator in the optimal system (5.4.3).

Example 5.5.1 *Reduction and invariant solution using X^6.*

Consider the generator $X^6 = X_1 + \alpha X_5 = \frac{\partial}{\partial t} - \alpha x \frac{\partial}{\partial \varphi} + \alpha \frac{\partial}{\partial \psi}$ where $\alpha \in \mathbb{R}$, from the optimal system (5.4.3). Solving the corresponding characteristic equations of the first prolongation

$$\frac{dt}{1} = \frac{dx}{0} = \frac{d\varphi}{-\alpha x} = \frac{d\psi}{\alpha} = \frac{d\varphi_t}{0} = \frac{d\varphi_x}{-\alpha} = \frac{d\psi_t}{0} = \frac{d\psi_x}{0},$$

gives the following invariants up to the first order

$$\begin{gathered} I_1 = x, \quad I_2 = \varphi + \alpha t x, \quad I_3 = \psi - \alpha t, \quad I_4 = \varphi_t, \\ I_5 = \varphi_x + \alpha t, \quad I_6 = \psi_t, \quad I_7 = \psi_x. \end{gathered} \tag{5.5.2}$$

Since the order of the Timoshenko system (5.1.1) is two, then the invariant conditions have to be functions of the first order invariants of the form

$$B_\Gamma(\ I_1,\ I_2,\ I_3,\ I_4,\ I_5,\ I_6,\ I_7\ ) = 0, \qquad \Gamma = 1, ..., 4. \tag{5.5.3}$$

The invariant conditions (5.5.3) are prescribed on the invariant surfaces

$$\omega_\gamma(I_1) = 0, \qquad \gamma = 1, 2, \tag{5.5.4}$$

which are invariants of order zero that depend on the independent variables.

For $\alpha \neq 0$, if we restrict the invariant conditions (5.5.3) to be linear then it will take the form

$$\begin{aligned} B_\Gamma = \; & A_{\Gamma,1}(x) + A_{\Gamma,2}(x)\ (\varphi + \alpha t x) + A_{\Gamma,3}(x)\ (\psi - \alpha t) + A_{\Gamma,4}(x)\ \varphi_t \\ & + A_{\Gamma,5}(x)\ (\varphi_x + \alpha t) + A_{\Gamma,6}(x)\ \psi_t + A_{\Gamma,7}(x)\ \psi_x = 0, \end{aligned} \tag{5.5.5}$$

where $A_{\Gamma,i}(x)$ are arbitrary functions for $\Gamma = 1, ..., 4$ and $i = 1, ..., 7$.
Moreover, the invariants of order zero I_1, I_2 and I_3 give the invariant variables

$$\varphi(t,x) \; = \; Z(\zeta) - \alpha\, t x, \quad \psi(t,x) \; = \; W(\zeta) + \alpha t, \quad \zeta \; = \; x. \tag{5.5.6}$$

The reduction of Timoshenko system (5.1.1) with the boundary conditions (5.5.5) prescribed on the surfaces $x = 0$ and $x = L$ using the invariant variables (5.5.6) is the system of ODEs of the form

$$\begin{aligned} & Z'' + W' = 0, \\ & EI\, W'' - k\ Z' - k\, W - \chi\,(\alpha) \; = \; 0, \end{aligned} \tag{5.5.7}$$

with general boundary conditions of the form

$$\begin{aligned}A_{\Gamma,1}(\zeta) + A_{\Gamma,2}(\zeta)\ Z + A_{\Gamma,3}(\zeta)\ W - \alpha A_{\Gamma,4}(\zeta)\ \zeta + A_{\Gamma,5}(\zeta)\ Z' \\ +\alpha A_{\Gamma,6}(\zeta) + A_{\Gamma,7}(\zeta)\ W' = 0,\end{aligned} \tag{5.5.8}$$

prescribed on the surfaces $\zeta = 0$ *and* $\zeta = L$.

For instance, if we consider a beam that models small motions of a hinged arm, which is hinged at the origin and free at its other end. This case was consider in [76] with two control functions. The control functions are the force $f(t)$ *applied at the free end, and a torque* $\tau(t)$ *applied at the hinged end. The associated boundary conditions*

$$\varphi(t,0) = 0, \quad \psi_x(t,0) = \tau(t), \quad \varphi_x(t,L) - \psi(t,L) = f(t), \quad \psi_x(t,L) = 0, \tag{5.5.9}$$

can be obtained from equation (5.5.5) for the non-zero values $A_{1,2} = 1$ *at* $x = 0$*;* $A_{2,1} = -r$, $A_{2,7} = 1$ *at* $x = 0$*;* $A_{3,1} = -q$, $A_{3,3} = -1$, $A_{3,5} = 1$ *at* $x = L$*;* $A_{4,7} = 1$ *at* $x = L$ *with force* $f(t) = q - 2\ \alpha t$ *and torque* $\tau(t) = r$.

Using the invariant variables (5.5.6), the boundary conditions (5.5.9) is reduced to

$$Z(0) = 0, \quad W'(0) = r, \quad Z'(L) - W(L) = q, \quad W'(L) = 0. \tag{5.5.10}$$

The solution of the boundary value problem (5.5.7) and (5.5.10) is given by

$$\begin{aligned}Z\,(\zeta) &= \tfrac{r}{6L}\zeta^3 - \tfrac{r}{2}\zeta^2 + \left(\tfrac{Lr}{2} - \tfrac{r\,EI}{2Lk} - \tfrac{\chi(\alpha)}{2k} + \tfrac{q}{2}\right)\zeta, \\ W\,(\zeta) &= -\tfrac{r}{2L}\zeta^2 + r\zeta - \tfrac{\chi(\alpha)}{2k} - \tfrac{1}{2}\left(q + r\left(\tfrac{EI}{Lk} + L\right)\right).\end{aligned} \tag{5.5.11}$$

Substituting back in the invariant variables (5.5.6), the system (5.1.1) with the boundary conditions (5.5.9) has the following solution:

$$\begin{aligned} \varphi\left(t,x\right) &= \tfrac{r}{6L}x^3 - \tfrac{r}{2}x^2 + \left(-\alpha\,t - \tfrac{\chi(\alpha)}{2k} - \tfrac{rEI}{2Lk} + \tfrac{q}{2} + \tfrac{rL}{2}\right)x, \\ \psi\left(t,x\right) &= \alpha\,t - \tfrac{r}{2L}x^2 + r\,x - \tfrac{\chi(\alpha)}{2k} - \tfrac{1}{2}\left(q + r\left(\tfrac{EI}{Lk} + L\right)\right). \end{aligned} \tag{5.5.12}$$

Table 5.2: Reductions using one-dimensional optimal system (5.4.3)

Generators in (5.4.3)	Invariant variables	The reduced system
$X^1 = X_1 + \alpha X_2 + \beta X_8$, $\alpha, \beta \neq 0$.	A	$(k-\alpha^2\rho_1)\,Z'' + k\,W' + \frac{\alpha\beta\,\rho_1 EI}{k}\zeta + \frac{1}{6}\left(\frac{\beta\,k}{\alpha} - \alpha\,\beta\,\rho_1\right)\zeta^3 = 0$, $(\rho_2\alpha^2 - EI)\;W'' + k\,Z' + k\,W + \chi\,(-\alpha\;W') - \frac{\beta\,EI}{2\alpha}\zeta^2 + \frac{\beta\,k}{24\alpha}\zeta^4 = 0$.
$X^2 = X_1 + \alpha X_2 + \beta X_4$, $\alpha \neq 0, \beta \in \mathbb{R}$.	B	$(k-\alpha^2\rho_1)\,Z'' + k\,W' - \beta\rho_1 = 0$, $(\rho_2\alpha^2 - EI)\,W'' - kZ' + k\,W + \chi\,(-\alpha\;W') = 0$.
$X^3 = X_1 + \alpha X_5 + \beta X_6 + \gamma X_8$, $\alpha, \beta \in \mathbb{R}, \gamma \neq 0$.	C	$Z'' + W' = 0$, $EI\,W'' - k\;Z' - k\,W - \chi\left(\frac{\gamma}{6}\,\zeta^3 + \beta\,\zeta + \alpha\right) = 0$.
$X^4 = X_1 + \alpha X_5 + \beta X_7$, $\alpha \in \mathbb{R}, \beta \neq 0$.	D	$Z'' + W' = 0$, $EI\,W'' - k\;Z' - k\,W - \chi\left(\alpha + \frac{\beta}{2}\,\zeta^2\right) = 0$.
$X^5 = X_1 + \alpha X_6$, $\alpha \neq 0$.	E	$Z'' + W' = 0$, $EI\,W'' - k\;Z' - k\,W - \chi\,(\alpha\,\zeta) = 0$.
$X^6 = X_1 + \alpha X_5$, $\alpha \in \mathbb{R}$.	F	$Z'' + W' = 0$, $EI\,W'' - k\;Z' - k\,W - \chi\,(\alpha) = 0$.
$X^7 = X_2 + \alpha X_8$, $\alpha \neq 0$.	G	$\rho_1\,Z'' = 0$, $\rho_2\,W'' + \chi\,(W') + k\,W + \frac{\alpha\,E\,I\,k}{2\,\rho_1}\zeta^2 = 0$.
$X^8 = X_2 + \alpha X_4$, $\alpha \in \mathbb{R}$.	H	$\rho_1\,Z'' = 0$, $\rho_2\,W'' + \chi\,(W') + k\,W + \alpha k\,\zeta = 0$.

$$
\begin{aligned}
A:\quad & \varphi(t,x) = Z(\zeta) - \tfrac{\beta}{24}\,tx^4 + \tfrac{\beta EI}{2k}tx^2 + \tfrac{\alpha\,\beta}{12}\,t^2x^3 - \tfrac{\alpha\,\beta EI}{2k}t^2x + \tfrac{\beta\,EI}{6\rho_1}t^3 - \tfrac{\alpha^2\beta}{12}x^2t^3 + \tfrac{\alpha^2\beta EI}{6k}t^3 \\
& + \tfrac{\alpha^3\beta}{24}\,xt^4 - \tfrac{\alpha^4}{120}\,t^5, \quad \psi(t,x) = W(\zeta) + \tfrac{\beta}{24\,\alpha}x^4, \quad \zeta = x - \alpha t. \\
B:\quad & \varphi(t,x) = Z(\zeta) + \tfrac{\beta}{2}t^2, \quad \psi(t,x) = W(\zeta), \quad \zeta(t,x) = x - \alpha t. \\
C:\quad & \varphi(t,x) = Z(\zeta) - \alpha\,tx - \tfrac{\beta}{2}\,tx^2 - \tfrac{\gamma}{24}\,tx^4 + \tfrac{\gamma EI}{2k}tx^2 + \tfrac{\gamma\,EI}{6\rho_1}t^3\,, \quad \psi(t,x) = W(\zeta) + \alpha t \\
& + \beta\,t\,x + \tfrac{\gamma}{6}\,t\,x^3, \quad \zeta = x. \\
D:\quad & \varphi(t,x) = Z(\zeta) + \left(\tfrac{\beta EI}{k} - \alpha\right)tx - \tfrac{\beta}{6}\,t\,x^3, \quad \psi(t,x) = W(\zeta) + \alpha\,t + \tfrac{\beta}{2}\,t\,x^2, \quad \zeta = x. \\
E:\quad & \varphi(t,x) = Z(\zeta) - \tfrac{\alpha}{2}x^2t, \quad \psi(t,x) = W(\zeta) + \alpha\,xt, \quad \zeta = x. \\
F:\quad & \varphi(t,x) = Z(\zeta) - \alpha\,tx, \;\; \psi(t,x) = W(\zeta) + \alpha t, \;\; \zeta = x. \\
G:\quad & \varphi(t,x) = Z(\zeta) + \tfrac{\alpha\,E\,I}{2\,\rho_1}\,t^2x + \tfrac{\alpha\,E\,I}{6\,k}x^3, \;\; \psi(t,x) = W(\zeta) + \tfrac{1}{24}\,\alpha\,x^4 - \tfrac{\alpha}{120}x^5, \;\; \zeta = t. \\
H:\quad & \varphi(t,x) = Z(\zeta) + \alpha\,tx, \;\; \psi(t,x) = W(\zeta), \;\; \zeta = t.
\end{aligned}
\tag{5.5.13}
$$

Table 5.3: Invariant conditions and theirs invariant reductions

X^i	Invariant condition	Invariant surface	Reduced invariant condition	Reduced invariant surface
X^1	$\hat{A}$	$\omega(x-\alpha\,t)$	$B\left(\zeta,\,Z,\,W,\,\frac{\beta EI}{2k}\zeta^2 - \frac{\beta}{24}\,\zeta^4 - \alpha Z',\,Z',\,-\alpha W',\,W'\right)$	$\omega(\zeta)$
X^2	$\hat{B}$	$\omega(x-\alpha\,t)$	$B\left(\zeta,\,Z,\,W,\,\alpha Z',\,Z',\,\alpha W',\,W'\right)$	$\omega(\zeta)$
X^3	$\hat{C}$	$\omega(x)$	$B\left(\zeta,\,Z,\,W,\,-\alpha\zeta - \left(\frac{\beta}{2} + \frac{\gamma EI}{2k}\right)\zeta^2 - \frac{\gamma}{24}\,\zeta^4,\,Z',\,\alpha + \beta\,\zeta + \frac{\gamma}{6}\,\zeta^3,\,W'\right)$	$\omega(\zeta)$
X^4	$\hat{D}$	$\omega(x)$	$B\left(\zeta,\,Z,\,W,\,\left(\frac{EI}{k}\beta - \alpha\right)\zeta - \frac{\beta}{6}\zeta^3,\,Z',\,\alpha + \frac{\beta}{2}\,\zeta^2,\,W'\right)$	$\omega(\zeta)$
X^5	$\hat{E}$	$\omega(x)$	$B\left(\zeta,\,Z,\,W,\,-\frac{\alpha}{2}\,\zeta^2,\,Z',\,\alpha\,\zeta,\,W'\right)$	$\omega(\zeta)$
X^6	$\hat{F}$	$\omega(x)$	$B(\zeta,\,Z,\,W,\,-\alpha\,\zeta,\,Z',\,\alpha,\,W')$	$\omega(\zeta)$
X^7	$\hat{G}$	$\omega\,(t)$	$B\left(\zeta,\,Z,\,W,\,Z',\,\frac{\alpha EI}{2\rho_1}\zeta^2,\,W',\,0\right)$	$\omega(\zeta)$
X^8	$\hat{H}$	$\omega\,(t)$	$B\left(\zeta,\,Z,\,W,\,Z',\,\alpha\,\zeta,\,W',\,0\right)$	$\omega(\zeta)$

$$
\begin{aligned}
\hat{A}: \quad & B\Big(x-\alpha\, t, \varphi+\tfrac{\alpha^4\beta}{120}\, t^5-\tfrac{\alpha^3\beta}{24}t^4x-\tfrac{\alpha^2\beta EI}{6k}t^3+\tfrac{\alpha^2\beta}{12}t^3x^2-\tfrac{\beta\, EI}{6\rho_1}t^3-\tfrac{\alpha\beta}{12}\, t^2x^3+\tfrac{\alpha\beta EI}{2k}t^2x+\tfrac{\beta}{24}\, tx^4-\tfrac{\beta EI}{2k}tx^2 \\
& ,\psi-\tfrac{\beta}{24\alpha}\, x^4, \varphi_t-\tfrac{\beta EI}{2\rho_1}\, t^2, \varphi_x-\tfrac{\beta\,\alpha^3}{24}t^4+\tfrac{\beta\,\alpha^2}{6}t^3x+\tfrac{\alpha\beta EI}{2k}t^2-\tfrac{\alpha\beta}{4}\, t^2x^2+\tfrac{\beta}{6}\, tx^3-\tfrac{\beta EI}{k}tx, \psi_t, \psi_x-\tfrac{\beta}{6\alpha}x^3\Big)\,. \\
\hat{B}: \quad & B\Big(x-\alpha\, t, \varphi-\tfrac{\beta}{2}\, t^2, \psi, \varphi_t-\beta t, \varphi_x, \psi_t, \psi_x\Big)\,. \\
\hat{C}: \quad & B\Big(x, \varphi+\alpha tx+\tfrac{\beta}{2}\, tx^2+\tfrac{\gamma}{24}tx^4-\tfrac{\gamma EI}{2k}tx^2-\tfrac{\gamma EI}{6\rho_1}t^3, \psi-\alpha t-\beta tx-\tfrac{\gamma}{6}tx^3, \varphi_t-\tfrac{\gamma EI}{2\rho_1}t^2, \varphi_x+\alpha t+\beta tx+\tfrac{\gamma}{6}tx^3 \\
& -\tfrac{\gamma\, EI}{k}tx, \psi_t, \psi_x-\beta t-\tfrac{\gamma}{2}\, tx^2\Big)\,. \\
\hat{D}: \quad & B\Big(x, \varphi+\alpha tx+\tfrac{\beta}{6}tx^3-\tfrac{\beta EI}{k}tx, \psi-\alpha\, t-\tfrac{\beta}{2}tx^2, \varphi_t, \varphi_x+\alpha\, t+\tfrac{\beta}{2}tx^2-\tfrac{\beta EI}{k}t, \psi_t, \psi_x-\beta tx\Big)\,. \\
\hat{E}: \quad & B\left(x, \varphi+\tfrac{\alpha}{2}tx^2, \psi-\alpha tx, \varphi_t, \varphi_x+\alpha tx, \psi_t, \psi_x-\alpha\, t\right)\,. \\
\hat{F}: \quad & B\left(x, \varphi+\alpha tx, \psi-\alpha t, \varphi_t, \varphi_x+\alpha t, \psi_t, \psi_x\right)\,. \\
\hat{G}: \quad & B\Big(t, \varphi+\tfrac{\alpha}{120}\, x^5-\tfrac{\alpha EI}{6k}x^3-\tfrac{\alpha EI}{2\rho_1}t^2x, \psi-\tfrac{\alpha}{24}\, x^4, \varphi_t-\tfrac{\alpha EI}{\rho_1}tx, \varphi_x+\tfrac{\alpha}{24}\, x^4-\tfrac{\alpha EI}{2k}x^2, \psi_t, \psi_x-\tfrac{\alpha}{6}\, x^3\Big)\,. \\
\hat{H}: \quad & B\left(t, \varphi-\alpha tx, \psi, \varphi_t-\alpha x, \varphi_x, \psi_t, \psi_x\right)\,.
\end{aligned}
\tag{5.5.14}
$$

5.6 Discussion and concluding remarks

Lie group study of a non-linear Timoshenko system of PDEs with frictional damping term in rotational angle is performed. Lie symmetry generators and their Lie group transformations for Timoshenko system are presented. A systematic approach and a formula for computing invariants in the adjoint representation is illustrated. Also the one-dimensional optimal system is derived for the corresponding Lie algebra. All possible invariant variables and their corresponding reductions for each vector field in the optimal system are found. The reductions to system of ODEs are given in Table 5.2. They are described by optimal reductions where all non-similar invariant solutions under symmetry transformations can be given from the solution of the reduced system of ODEs. Hinged-Free beam with two control functions: constant torque applied at the hinged end, and linear force applied at the free end. Finally, all possible non-similar invariant conditions prescribed on in invariant surfaces under symmetry transformations are given in Table 5.3.

CHAPTER 6

CONCLUSIONS AND FUTURE WORK

Conclusions

In this dissertation we performed Lie symmetry group analysis to classify, reduce, derive one-dimensional optimal system and find invariant solutions of two nonlinear Timoshenko beam models arising in many engineering applications.

In the linear case, the Lie symmetry group analysis does not yield enough infinitesimals and in turn give only trivial group analysis. This forced us to look for another technique that may serve our need to find the exact solution of such a problem. This technique (based on finite Fourier sine and cosine transformations) is used to find the exact solution of a linear hinged ends beam in Timoshenko model. This solution is used to show the effect of the material properties by considering two different beam materials (Aluminum and Polycarbonate). The behavior of the displacement and rotational angle of the two materials are presented using graphs for different space and time values.

In chapter 2, the analytical solution of the Timoshenko beam system with the appropriate boundary conditions is presented. The system of equations is decoupled prior to finding solutions, and the boundary conditions for the transverse displacement $\varphi(x, t)$ and the angular rotation are introduced in the form of periodic functions. The new initial-boundary value problem has been solved after incorporating the finite Fourier sine transformation. The response of the Timoshenko beam system has been formulated in terms of the rotational angle $\psi(x, t)$. Later, the finite Fourier cosine transformation has been used to obtain the closed form solution. The solutions for displacement $\varphi(x, t)$ and rotational angle $\psi(x, t)$ for the first and second problems are verified to make sure that these are solutions of the original system.

In chapter 3, analytical solution for the Timoshenko system with hinged-hinged and two weak damping configuration is presented. External excitation for flexural and torsional oscillations is introduced to assess the dynamic response of the beam. The dimension of the beam cross-section is varied with changing the aspect ratio of the cross-section (width and height are varied) while keeping the cross-sectional area constant. This arrangement enables to examine the effect of beam size on the flexural and torsional characteristics of the beam. Two beam materials are incorporated in the simulations, namely, concrete and steel. It is found that without introducing damping, flexural and torsional oscillations follow the elastic behavior; in which case, the natural frequency and the oscillation amplitude remain high, which is more pronounced for steel beam. Introducing damping in the flexural motion, while considering elastic behavior for torsional motion, results in logarithmic amplitude decay for flexural and torsional oscillations. The coupling of governing equations of motion is responsible for the amplitude decay in torsional oscillation despite the damping coefficient is set to zero for the torsional motion. The logarithmic decay of the amplitude in both flexural and torsional oscillations is more pronounced for concrete beam. This behavior is associated with the larger damping coefficient of concrete beam than that corresponding to steel beam. The damping frequency of the flexural and the torsional oscillations is also lower for concrete beam than that of steel beam. Consequently, steel beam undergoes high cyclic oscillation than concrete beam, which indicates that steel bar is more prone to the high cyclic fatigue than concrete beam, i.e. under same conditions of external excitation for flexural and torsional oscillations; steel beam suffers from high cyclic oscillation and resulting fatigue. The effect of size of the beam cross-section on the flexural and the torsional oscillations is significant for concrete and steel beams. In this case, the damping factor of beam changes due to the change of second moment of area, which is used for calculations of the damping factor, i.e. reducing damping factor, due to the change of beam cross-sectional size, gives rise to increased amplitude and frequency of the oscillation of the beam regardless of the beam material. The present study provides useful information on the dynamic response of hinged-hinged beam, resembling actual structures, when externally excited and enhances the understanding of the effect of beam size and material on the flexural and the torsional response of beam.

In chapter 4, the complete Lie symmetry classification of a non-linear Timoshenko system of PDEs with frictional damping term in rotational angle is performed. The classification is related to the arbitrary dependence on the rotation moment $\chi(\psi(x))$. A Lie symmetry analysis is performed in three cases for non-linear rotational moment. The three cases depend on the sign of the parameter $d^2 - 4k\rho_2$. The one-dimensional optimal system is derived for each one of the three cases. All possible invariant forms and their corresponding reductions for each vector field in the optimal systems are found. These reductions to systems of ODEs are given. They are described by optimal reduction where all non-similar invariant solutions under symmetry transformations can be given from the solution of these reduced system of ODEs.

In chapter 5, a Lie group study of a non-linear Timoshenko system of PDEs with frictional damping term in rotational angle is performed. Lie symmetry generators and their Lie group transformations for Timoshenko system are presented. A systematic approach and a formula for computing invariants in the adjoint representation is illustrated. Also the one-dimensional optimal system is derived for the corresponding Lie algebra. All possible invariant variables and their corresponding reductions for each vector field in the optimal system are found. The reductions to system of ODEs are given. They are described by optimal reductions where all non-similar invariant solutions under symmetry transformations can be obtained from the solution of the reduced system of ODEs. Hinged-Free beam is considered as an example with two control functions: constant torque applied at the hinged end, and linear force applied at the free end. Finally, all possible non-similar invariant conditions prescribed on invariant surfaces under symmetry transformations are also given.

Future work

This study opens the prospect to many possible future research work such as:

1. Lie infinitesimal generators in the linear case were not sufficient to give any significant exact solutions. This suggest to look for higher-order symmetries such as the contact and the Lie-B$\ddot{a}$cklund symmetries.

2. In this work we study linear Timoshenko beam model using finite Fourier transform without any applied load source function. In the next step, we can add arbitrary source function of the space variable x or even a function of the displacement $\varphi(x, t)$.

3. Different type of boundary conditions like Clamped-Clamped, Clamped-Free, ..., can be considered with or without source function with suitable initial conditions.

4. Different type of nonlinear damping can be assumed other than the weak damping which is considered in this dissertation.

5. Higher-order contact or Lie-Bäcklund symmetries can be found to investigate if more symmetries arise which in turn may give more significant exact solutions for the nonlinear model.

6. Studying Timoshenko linear and nonlinear beam theory opens the direction to similar study from Lie symmetry point of view of the problem arising in elastic plate theory.

REFERENCES

[1] Abbas B and Thomas J, *The second frequency spectrum of Timoshenko beams*, Journal of Sound and vibration 51, no. 1 (1977): 123-137.

[2] Azad H, Biswas I, Ghanam R and Mustafa T, *On computing joint invariants of vector fields*. Journal of Geometry and Physics 97 (2015): 69-76.

[3] Baumann G, *Symmetry Analysis of Differential Equations With Mathematica*. Springer Science and Business Media, 2013.

[4] Blinkov, Yuri A, Carlos F, Vladimir P, Plesken W and Robertz D, *The MAPLE package Janet: II. linear partial differential equations*. In Proceedings of the 6th International Workshop on Computer Algebra in Scientific Computing, 41-54. 2003.

[5] Bluman G and Anco S, *Symmetry and Integration Methods for Differential Equations*. Vol. 154. Springer Science and Business Media, 2008.

[6] Bluman G and Kumei S, *Symmetries and Differential Equations*. Vol. 154. Springer Science and Business Media, 2013.

[7] Bokhari A H, Mahomed F M and Zaman F D, *Symmetries and integrability of a fourth-order EulerBernoulli beam equation.* Journal of Mathematical Physics 51, no. 5 (2010): 053517.

[8] Butterworth J, Lee J H and Davidson B, *Experimental determination of modal damping from full scale testing*, In 13th world conference on earthquake engineering, Vancouver, Paper, vol. 310. 2004.

[9] Capsoni A, Maria Vigano G and Bani-Hani K, *On damping effects in Timoshenko beams.* Int. J. of Mechanical Sciences, Vol. 73, (2013): 27-39.

[10] Carminati J, John S and Greg J, *Isogroups of differential equations using algebraic computing*, Journal of Symbolic Computation 14, no. 1 (1992): 103-120.

[11] Chen W-R, *Parametric studies on bending vibration of axially-loaded twisted Timoshenko beams with locally distributed Kelvin-Voigt damping*, Int. J. of Mechanical Sciences, Vol. 88, (2014): 61-70.

[12] Chen T, Su G Y, Shen Y S, Gao B, Li X Y and Muller R, *Unified Green's functions of forced vibration of axially loaded Timoshenko beam: Transition parameter.* Int. J. of Mechanical Sciences, Vol. 113, (2016): 211-220.

[13] Chupakhin A P, *Optimal systems of subalgebras of one Solvable Algebra L7.* Novosibirsk, Russia, RFFR-project (1994): 93-013.

[14] Dehrouyeh-Semnani A M and Bahrami A, *n size-dependent Timoshenko beam element based on modified couple stress theory*, Int. J. of Engineering Science, Vol. 107, (2016): 134-148.

[15] Davies R M, *LV. The frequency of transverse vibration of a loaded fixed-free bar.III. The effect of the rotatory inertia of the bar*. The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science 23, no. 155 (1937): 563-573.

[16] Debnath L and Bhatta D, *Integral Transforms and Their Applications*. CRC press, 2014.

[17] Djondjorov P A, *Invariant properties of Timoshenko beam equations*. International journal of engineering science 33, no. 14 (1995): 2103-2114.

[18] Dimas S and Dimitri T, *SYM: A new symmetry-finding package for Mathematica*. In Proceedings of the 10th international conference in modern group analysis, (2004): 64-70.

[19] Ding L, Zhu H-P and Wu L, *Effects of axial load and structural damping on wave propagation in periodic Timoshenko beams on elastic foundations under moving loads*, Physics Letters A, Vol.e 380, Issues 31-32, (2016): 2335-2341.

[20] Dolph C L, *On the Timoshenko theory of transverse beam vibrations*, Quarterly of Applied Mathematics, Vol. 12, no. 2, (1954): 175-187.

[21] Ecsedi I and Baksa A, *Analytical solution for layered composite beams with partial shear interaction based on Timoshenko beam theory.* Engineering Structures, Vol. 115, (2016): 107-117.

[22] Failla G, *An exact generalised function approach to frequency response analysis of beams and plane frames with the inclusion of viscoelastic damping*, J. of Sound and Vibration, Vol. 360, (2016): 171-202.

[23] Fatiha A, *Strong lower energy estimates for nonlinearly damped Timoshenko beams and Petrowsky equations.* Nonlinear Differential Equations and Applications NoDEA 18, no. 5 (2011): 571-597.

[24] Fatima A, Bokhari A H, Mahomed F M and Zaman F D, *A note on the integrability of a remarkable static EulerBernoulli beam equation.* Journal of Engineering Mathematics 82, no. 1 (2013): 101-108.

[25] Galas F and Richter E W, *Exact similarity solutions of ideal MHD equations for plane motions.* Physica D: Nonlinear Phenomena 50, (1991): 297-307.

[26] Gu L and Qin Z, *Chu, Analytical analysis of the thermal effect on vibrations of a damped Timoshenko beam*, Mechanical Systems and Signal Processing, Vol. 60-61, (2015): 619-643.

[27] Han H, Cao D and Liu L, *Green's functions for forced vibration analysis of bending-torsion coupled Timoshenko beam*, Applied Mathematical Modelling, Vol. 45, (2017): 621-635.

[28] Han S M, Benaroya H and Wei T, *Dynamics of transversely vibrating beams using four engineering theories*. Journal of Sound and vibration 225, no. 5 (1999): 935-988.

[29] Heck, Andre and Wolfram K, *Introduction to MAPLE*. Vol. 1993. New York: Springer-Verlag, 1993.

[30] Huang Y, Zhang M and Rong H, *Buckling Analysis of Axially Functionally Graded and Non-Uniform Beams Based on Timoshenko Theory*. Acta Mechanica Solida Sinica, Vol. 29, Issue 2, (2016): 200-207.

[31] Huang T C, *The effect of rotatory inertia and of shear deformation on the frequency and normal mode equations of uniform beams with simple end conditions*. Journal of Applied Mechanics 28, no. 4 (1961): 579-584.

[32] Hu X, Li Y and Chen Y, *A direct algorithm of one-dimensional optimal system for the group invariant solutions*. Journal of Mathematical Physics 56, no. 5 (2015): 053504.

[33] Hydon P E, *Symmetry Methods for Differential Equations*. Vol. 22. Cambridge University Press, 2000.

[34] Ibragimov N H, *Elementary Lie Group Analysis and Ordinary Differential Equations*. Vol. 197. Chichester: Wiley, 1999.

[35] Ibragimov N H, *CRC Handbook of Lie Group Analysis of Differential Equations*. Vol. 3. CRC press, 1995.

[36] Ibragimov N H, *Sophus Lie and harmony in mathematical physics, on the 150th anniversary of his birth.* The Mathematical Intelligencer 16, no. 1 (1994): 20-28.

[37] Ibragimov N H, Torrisi M and Valenti A, *Preliminary group classification of equations* $v_{tt} = f(x, v_x)v_{xx} + g(x, v_x)$. Journal of Mathematical Physics 32, no. 11 (1991): 2988-2995.

[38] Ibragimov N H, *Selected Works, Vol. II.* ALGA publ., BTH, Karlskrona, Sweden, 2006.

[39] Kahrobaiyan M H, Asghari M and Ahmadian M T, *A Timoshenko beam element based on the modified couple stress theory.* Int. J. of Mechanical Sciences, Vol. 79, (2014): 75-83.

[40] Kotz H, *A Technique to Classify the Similarity Solutions of Nonlinear Partial (Integro-) Differential Equations. I. Optimal Systems of Solvable Lie Subalgebras.* Zeitschrift fr Naturforschung A 47, no. 11 (1992): 1161-1174.

[41] Kruszewski E T, *Effect of Transverse Shear and Rotary Inertia on the Natural Frequency of a Uniform Beam*, 1949.

[42] Lanzoni L and Radi E, *A loaded Timoshenko beam bonded to an elastic half plane.* Int. J. of Solids and Structures, Vol. 92-93, (2016): 76-90.

[43] Lie S, *Theorie Der Transformationsgruppen.* Vol. 1. Teubner: Leipzig, Germany, 1888.

[44] Lie S, *Vorlesungen über Differentialgleichungen mit Bekannten Infinitesimalen Transformationen*. BG Teubner, 1891.

[45] Metsebo J, Nana N and Woafo P, *Dynamic responses of a hinged-hinged Timoshenko beam with or without a damage subject to blast loading*, Mechanics Research Communications, Vol. 71, (2016): 38-43.

[46] Moghaddas M, Esmailzadeh E, Sedaghati R and Khosravi P, *Vibration control of Timoshenko beam traversed by moving vehicle using optimized tuned mass damper*, J.of Vibration and Control, Vol. 18(6), (2011): 757-773.

[47] Mustafa M I and Messaoudi S A, *General energy decay rates for a weakly damped Timoshenko system*. Journal of Dynamical and Control Systems 16, no. 2 (2010): 211-226.

[48] Naz R and Mahomed F M, *Dynamic Euler-Bernoulli Beam Equation: Classification and Reductions*. Mathematical Problems in Engineering, (2015).

[49] O'Connor J and Robertson E F, *MacTutor History of Mathematics Archive*, University of St Andrews, Scotland, 2003.

[50] Oliveri F, *Lie symmetries of differential equations: classical results and recent contributions*. Symmetry 2, no. 2 (2010): 658-706.

[51] Olver P J, *Applications of Lie Groups to Differential Equations*. Vol. 107. Springer Science and Business Media, 2000.

[52] Olver P J, *Classical Invariant Theory.* Vol. 44. Cambridge University Press, 1999.

[53] Ovsiannikov L V, *Group Analysis of Differential Equations*, Academic Press, New York, 1982.

[54] Ovsiannikov L V, *On Optimal Systems of Subalgebras, Lie Groups and Their Applivations*, 1, 1994.

[55] Ovsiannikov L V, *Some results of the implementation of the PODMODELI program for the gas dynamics equations.* Journal of Applied Mathematics and Mechanics 63, no. 3 (1999): 349-358.

[56] Ozkaya E and Pakdemirli M, *Group-theoretic approach to axially accelerating beam problem.* Acta. Mech, 155, (2002): 111-123.

[57] Parayil D V, Kulkarni S S and Pawaskar D N, *Analytical and numerical solutions for thick beams with thermoelastic damping*, Int. J. of Mechanical Sciences, Vol. 94-95, (2015): 10-19.

[58] Patera J, Winternitz P and Zassenhaus H, *Continuous subgroups of the fundamental groups of physics. I General method and the Poincare group*, J. Math. Phys. (1975): 1597-1614.

[59] Paul A and Das D, *Non-linear thermal post-buckling analysis of FGM Timoshenko beam under non-uniform temperature rise across thickness.* Engineering Science and Technology, an Int. J., Vol. 19, Issue 3, (2016): 1608-1625.

[60] Pei P, *Stability of Mindlin-Timoshenko plate with nonlinear boundary damping and boundary sources*, J. of Mathematical Analysis and Applications, Vol. 448, Issue 2, (2017): 1467-1488.

[61] Rees E L, *Graphical discussion of the roots of a quartic equation*, The American Mathematical Monthly, Vol. 29, no. 2, (1922): 51-55.

[62] Rivera J and Racke R, *Timoshenko systems with indefinite damping*, J. of Mathematical Analysis and Applications, Vol. 341, Issue 2, (2008): 1068-1083.

[63] Rivera M, Jaime E and Racke R, *Global stability for damped Timoshenko systems*, 2002.

[64] Rivera J E and Reinhard R, *Timoshenko systems with indefinite damping.* J. of Mathematical Analysis and Applications, Vol. 341, no. 2, (2008): 1068-1083.

[65] Rocha F, Tarcsio M and Annibal F, *[SADE] a Maple package for the symmetry analysis of differential equations.* Computer Physics Communications 182, no. 2 (2011): 467-476.

[66] Roux A, Merwe A and van Rensburg N, *Elastic waves in a Timoshenko beam with boundary damping*, Wave Motion, Vol. 57, (2015): 194-206.

[67] Schwarz F, *Symmetries of differential equations: from sophus Lie to computer algebra.* Siam Review 30, no. 3 (1988): 450-481.

[68] Simsek M, *Non-linear vibration analysis of a functionally graded Timoshenko beam under action of a moving harmonic load*. Composite Structures, Vol. 92, Issue 10, (2010): 2532-2546.

[69] Singh B and Nanda B K, *Dynamic analysis of damping in layered and welded beams*, Engineering Structures, Vol. 48, (2013): 10-20.

[70] Soh C W, *EulerBernoulli beams from a symmetry standpoint-characterization of equivalent equations*. Journal of Mathematical Analysis and Applications 345, no. 1 (2008): 387-395.

[71] Soufyane A, *Stabilisation de la poutre de Timoshenko*. Comptes Rendus de l'Acadmie des Sciences-Series I-Mathematics 328, no. 8 (1999): 731-734.

[72] Stephani H and Malcolm M, *Differential Equations: Their Solution Using Symmetries*. Cambridge University Press, 1989.

[73] Strutt J W, *The Theory of Sound. Vol. 1.* (1877).

[74] Strutt J W and Rayleigh B, *The Theory of Sound, vol. 1*, 1945.

[75] Tatar N, *Exponential decay for a viscoelastically damped timoshenko beam*, Acta Mathematica Scientia, Vol. 33, Issue 2, (2013): 505-524.

[76] Taylor S W, *A smoothing property of a hyperbolic system and boundary controllability*. Journal of computational and applied mathematics 1 (2000): 23-40.

[77] Tebou L, *Indirect stabilization of a Mindlin-Timoshenko plate*, J. of Mathematical Analysis and Applications, Vol. 449, Issue 2, (2017): 1880-1891.

[78] Timoshenko S, *History of Strength of Materials: With a Brief Account of the History of Theory of Elasticity and Theory of Structures*. Courier Corporation, 1953.

[79] Timoshenko S P, *On the correction for shear of the differential equation for transverse vibrations of prismatic bars*. The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science 41, no. 245 (1921): 744-746.

[80] Timoshenko S P, *On the transverse vibrations of bars of uniform cross-section*. The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science 43, no. 253 (1922): 125-131.

[81] Traill-Nash R W and Collar A R, *The effects of shear flexibility and rotatory inertia on the bending vibrations of beams*. The Quarterly Journal of Mechanics and Applied Mathematics 6, no. 2 (1953): 186-222.

[82] Turnbull H, *Theory of Equations*, Oliver and Boyd, Edinburgh, UK, 1963.

[83] Ugural A C and Fenster S K. *Advanced Strength and Applied Elasticity*. Elsevier Publishing Company, 1981.

[84] Vassilev V M and Djondjorov P A, *Symmetry Groups and Conservation Laws in Structural Mechanics*. In Proceedings of Institute of Mathematics of NAS of Ukraine, vol. 30, no. Part 1, (2000): 223-230.

[85] Xiaorui H and Li Y and Chen Y, *A direct algorithm of one-dimensional optimal system for the group invariant solutions*, Journal of Mathematical Physics 56, (2015): 053504.

[86] Yaglom and Klein I M and Lie S, *Evolution of the Idea of Symmetry in the Nineteenth Century*. Birkhauser: Boston, MA, USA, 1988.

[87] Zhang J, Perez R J and Lavernia E J, *Documentation of damping capacity of metallic, ceramic and metal-matrix composite materials*. Journal of Materials Science 28, no. 9 (1993): 2395-2404.

[88] Zhang Z, Huang X, Zhang Z and Hua H, *On the transverse vibration of Timoshenko double-beam systems coupled with various discontinuities*, Int. J. of Mechanical Sciences, Vol. 89, (2014): 222-241.

Vitae

- Name: Shadi Mohammad Al-Omari
- Nationality: Jordan
- Date of Birth: September 21, 1983
- Email: *shalomari@kfupm.edu.sa*
- Permenant Address: Amman, Jordan

⋆ B.Sc. in Mathematics, Al-Hussein Bin Talal University, Ma'an, Jordan (2001 – 2004).

⋆ M.Sc. in Mathematics, Muatah University, Karak, Jordan (2005 – 2008).

⋆ Ph.D in Mathematics, King Fahd University of Petroleum and Minerals, Dhahran, Saudi Arabia (2012 – 2016).

- Publications: Al-Omari, S.; Zaman, F.; Azad, H. Lie Symmetries, Optimal System and Invariant Reductions to a Nonlinear Timoshenko System. Mathematics 2017, 5-34.